BASIC TRIGONOMETRY

C. O. TUCKEY

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BASIC TRIGONOMETRY

By

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PREFACE

In this book trigonometry begins with the sine, the other ratios being introduced only as they are needed. The sine is much the most important ratio for linking the new subject at an early stage with easy geometrical theorems, and its graph is the most valuable for those interested in applications of electricity involving periodicity. There is now general agreement that the start should be made with a single ratio, either sine or tangent; the latter provides just as easy a first stage, but the second stage becomes harder. The old method of flinging all six ratios at once at the pupil, leaving him to disentangle them by means of mnemonics and frequent drill, has little to recommend it. The symmetry of the six ratios is superficial: each has its special properties and uses, *e.g.*, the cosine for projections, and these are emphasised when the ratios are introduced separately.

The sine is defined at the start for obtuse as well as acute angles—"the sine gives the height." This makes it possible to reach important practical problems, using the sine formula, much more quickly than if the tangent is given the first place.

For those who begin trigonometry at an early age a four-figure table of sines without the difference columns is provided. This will suffice for Chapter I and nearly all of Chapter II, after which the usual four-figure tables will be needed.

It is claimed that, starting with the sine, more real trigonometry is covered in a smaller space. All the ratios and their special uses are however dealt with in the first five chapters, and any teacher who feels that the tangent has been unduly postponed can take the preliminary pages of Chapter IV earlier without detriment to the sequence.

An endeavour has been made throughout to select examples in close touch with reality, and users of this book will not remain unaware that they live in a three-dimensional world.

No apology is needed in these days for the introduction of triangles of velocities or even of navigation on a sphere.

Instead of the usual final " Revision Papers " or " Miscellaneous Examples " (which the teacher so often replaced by collections of the appropriate examination papers), a series of over thirty " Topics of Trigonometrical Interest " has been added. These are drawn from many fields and are of varied length and difficulty, providing tasks for all for odd times as well as something to stretch the more promising pupils.

Although it covers the trigonometry needed for Advanced Level Mathematics of the General Certificate of Education, this book has been written primarily as an introduction to the study of trigonometry. The tests for the inclusion or rejection of any item have been its importance in the development of the subject, its usefulness or its intrinsic interest.

It remains to add that the warmest thanks are due to Mr. A. P. Rollett, formerly senior mathematical master at Sevenoaks School, for the immense amount of work which he contributed to the making of this book, especially in adding variety to the examples.

C. O. T.

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BASIC TRIGONOMETRY

CHAPTER I

THE SINE OF AN ANGLE

CONSIDER this problem :

A ladder 10 *ft. long makes an angle of* 53° *with the ground and rests against a tall wall. How high up the wall does it reach ?*

This can be solved by scale-drawing and measurement, and the answer is almost exactly 8 ft., which is 0·8 of the length of the ladder. If we were now asked " how high would a 20ft. ladder reach, *at the same angle*," we should not have to make another drawing. For it is easy to see (from the same drawing) that the 20-ft. ladder would reach twice as high as the 10-ft. ladder, *i.e.*, 16 ft., which is 0·8 of 20 ft. Similarly, a 12-ft. ladder would reach a height of 0·8 of 12 ft., or 9·6 ft.

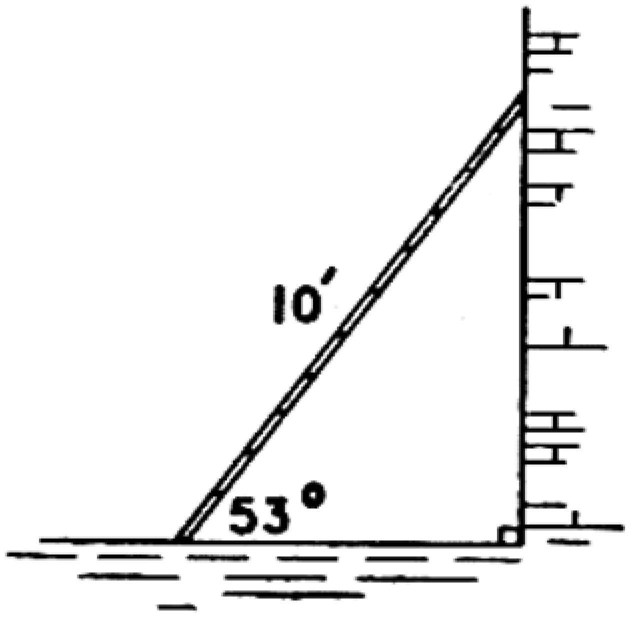

FIG. 1.

The fraction (or ratio) $\frac{\text{height reached}}{\text{length of ladder}}$ *depends only on the angle.*

This ratio is called the *sine* of the angle. Its value has been worked out for a large number of angles and the results recorded in tables. From these, many problems can be solved by calculation much more accurately than by drawing. It is the use of names for ratios of this kind and of tables of the values of these ratios which is the mark of trigonometry.

A fuller explanation of the meaning of the sine, applying to both acute and obtuse angles, is given on the next page.

The Sine of an Angle

In Figs. 2 and 3 an angle AOP is formed by turning a radius of a circle from the position OA to the position OP. In Fig. 2 the angle is acute, in Fig. 3 it is obtuse. In each case PN is drawn perpendicular to the diameter AB, which is thought of as horizontal.

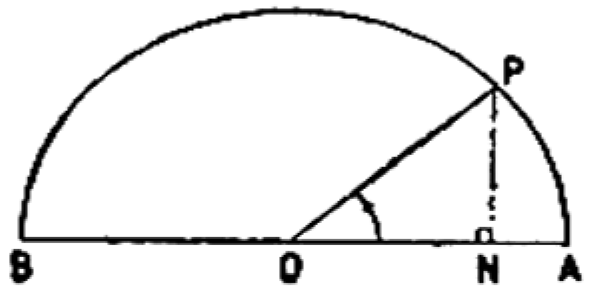

FIG. 2.

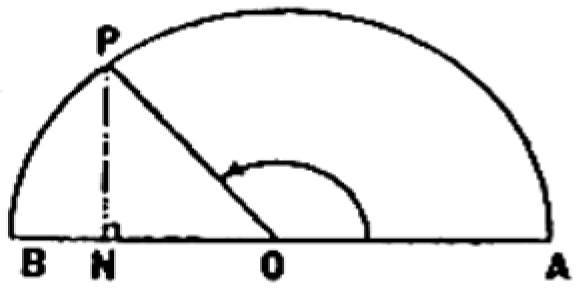

FIG. 3.

Then in both cases the ratio $\frac{PN}{OP}$ is called the sine of the angle AOP, so that *the sine gives the height of P above* AB *as compared with the radius of the circle.*

If, in Fig. 2, the radius turns the other way from the horizontal OB to the position OP through the obtuse angle BOP, the height above AB is PN as before, and the sine of angle BOP is $\frac{PN}{OP}$. Since $\angle BOP = 180° - \angle AOP$, this means that

$$\text{the sine of } (180° - \angle AOP) = \text{the sine of } \angle AOP.$$

This is also true in Fig. 3. The sines of the angles AOP and BOP are equal.

Notation

The sine of $\angle$AOP is written sin AOP for short (though pronounced sine AOP). Various single letters are often used to denote angles, especially the Greek letters

α (alpha), β (beta), γ (gamma), δ (delta), π (pi),
θ (theta), ϕ (phi), λ (lambda), μ (mu), ω (omega).

Suppose that, in either Fig. 2 or Fig. 3, $\angle$AOP is called θ, then $\angle$BOP is $(180° - \theta)$, and whether θ is acute or obtuse,

$$\sin (180° - \theta) = \sin \theta.$$

As examples of this, $\sin 150° = \sin 30°$, $\sin 145° = \sin 35°$.

Alternative Definition

Another way of explaining what the sine of an angle means is given on p. 7. This explanation however fits only acute angles, and if you depend on it you must just *learn* sin (obtuse angle) = sin (supplement) as a working rule, the supplement of θ being $(180° - \theta)$.

Examples—1

Finding Sines from Diagrams

1. If, in Fig. 4, the radius of the circle is 3 in. and P is at a height 2 in. above AB, what is the sine of $\angle AOP$? Make a sketch showing the two possible positions of P.

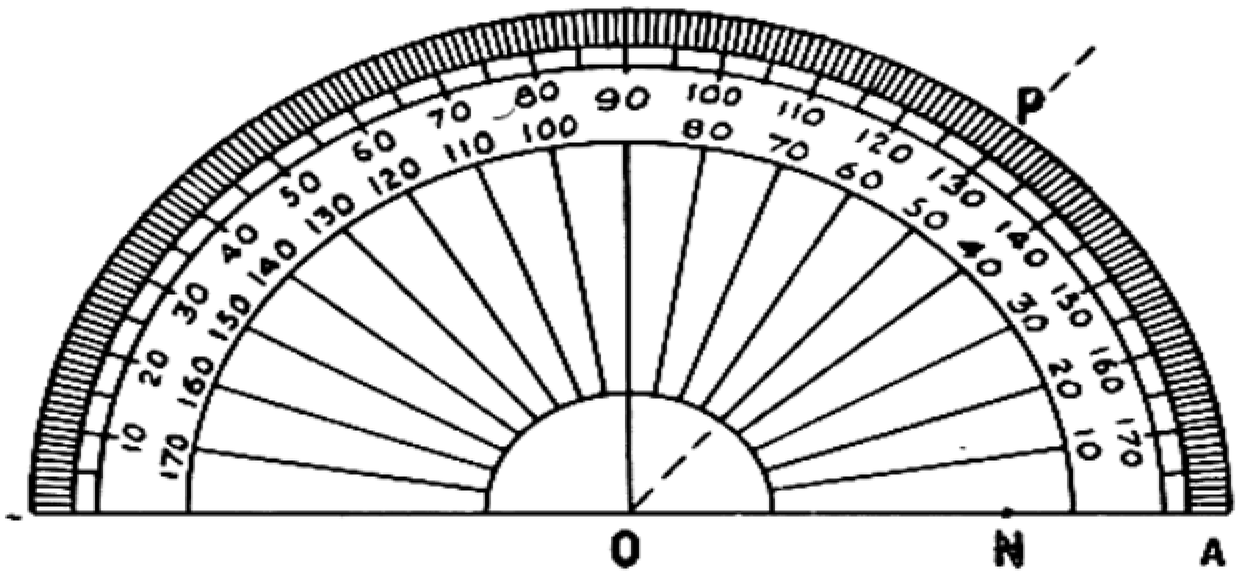

FIG. 4.

[Sol'n. The height is 2 in. and the radius 3 in., so sin AOP = $\frac{2}{3}$.]

2. Fig. 4 shows a semicircular protractor of a common type, except that letters A, O, B, P, N have been added.

Measurement shows that OA = OP = 2 in., PN = 1·53 in. What value does this give for sin 50° and sin 130° ?

3. Without *drawing* any of the perpendiculars, measure, on your own protractor, the perpendiculars to the diameter AB (from the 0° to the 180° mark) from the various points P marked 10°, 20°, 30° . . . and make a table of the sines of these angles up to 170°. Compare with the table of sines. [Why does the printed table only go as far as 90° ?]

4. Your table in No. 3 would be more complete if it began with 0° and ended with 180°. What is the height above AB of a point at A or B ? Can you guess sin 0° ? If not, see table.

5. If OP is the radius at 90° to AB, what is the height of P above AB compared with the radius ? Explain why sin 90° = 1.

6. In a figure like Fig. 2 draw OC the radius perpendicular to OA and let OPC be an equilateral triangle drawn on OC on the same side as A. How many degrees are there in $\angle$AOP ? Why is PN half the radius ? [This shows that $\sin 30^\circ = \frac{1}{2}$.]

7. If a 12-foot ladder rests against a wall at a height of 10 ft. above the ground, what is the sine of the angle which it makes with the ground ?

Finding Sines from the Tables

A right angle is 90 degrees. Each degree is divided into 60 *minutes*. This is written $1^\circ = 60'$, and so $0{\cdot}1^\circ = 6'$.

The table printed on pp. 200, 201, gives the sines of angles at intervals of six minutes or a tenth of a degree, and it is quite a straightforward matter to find them. Note that the decimal point is not printed, but is understood throughout in front of the printed figures.

8. Find sin 14°, sin 73°24′, sin 46°54′, sin 42°30′, sin 15·2°.

[Specimen sol'n : $\sin 83^\circ 36' = {\cdot}9938$, $\sin 26{\cdot}2^\circ = {\cdot}4415$.]

9. Remembering that for obtuse angles you must first subtract from 180°, find sin 126°, sin 100°30′, sin 98°54′, sin 153·7°.

[Specimen sol'n :
$\sin 134^\circ = \sin 46^\circ = {\cdot}7193$
$\sin 129^\circ 48' = \sin 50^\circ 12' = {\cdot}7683 \quad (180^\circ - 129^\circ 48' = 50^\circ 12')$
$\sin 96{\cdot}5^\circ = \sin 83{\cdot}5^\circ = {\cdot}9938 \quad (180^\circ - 96{\cdot}5^\circ = 83{\cdot}5^\circ)$.]

Using Sines. Draw a figure for each example.

10. In Fig. 5, P is a balloon held by a steel cable OP which is 3,000 ft. long. How high is P above the ground OA if $\angle AOP = 57^\circ$?

[Sol'n. If PN is perpendicular to OA, then $\frac{PN}{OP} = \sin 57^\circ$, $\therefore \frac{PN}{OP} = {\cdot}8387$ by the table.

$\therefore PN = OP \times {\cdot}8387 = 3000 \times {\cdot}8378$
$\bumpeq 2516$ ft., or, to three figures, 2520 ft.

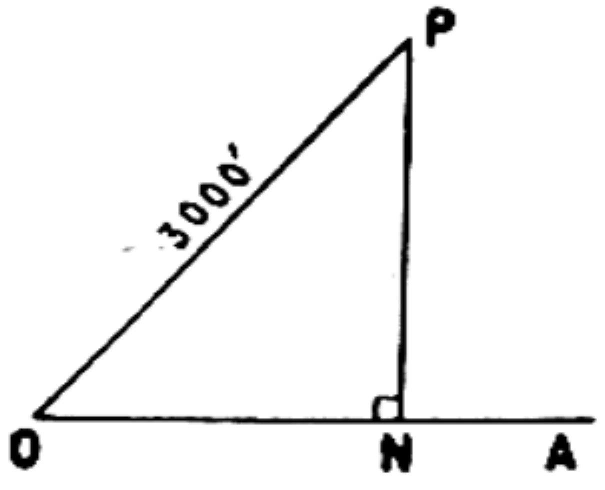

FIG 5.

You must never give more significant figures in your answer than four with four-figure tables or five with five-figure tables, and the final answer is often given with one figure less.

11. A ladder, 20 ft. long, rests against a wall so that it makes 64° with the ground. How high up the wall does it reach, to the nearest foot ?

12. In a conical tent the length of the canvas from the tod of the pole to the ground is 11 ft., and this canvas is inclined at 58° to the ground. How high is the pole ?

13. An aeroplane leaves the ground and climbs at a steady angle of 5° to the horizontal. When it has gone 2000 ft., how high is it ? When it has gone for one minute, after leaving the ground, at 240 m.p.h. (which is the same as 352 ft. per second) how high is it ?

14. A road slopes upward at 11° to the horizontal. If I walk 250 yds. along it, how many feet have I risen vertically ?

15. The beam of a searchlight is visible for 2500 ft. It slants at 65° to the horizontal. How high is the tip of the visible part of the beam ?

16. John walks $\frac{1}{4}$ mile along a road which makes 27° with the line due North from him (and 63° with the line due East). How far is he north of his starting point ?

17. The sloping side of a railway embankment is 50 ft. long, going straight up the slope, and it makes $38\frac{1}{2}°$ with the horizontal. How high is the top of the embankment above the base ?

18. A toboggan slides down a slope 250 ft. long which makes 23·2° with the horizontal. How far has it come down vertically ?

Finding Angles from Sines. [For diagrams use squared paper if available.]

19. Take AB = 6 in. and draw a semicircle on diameter AB with centre O. Draw a line 2 in. from AB and parallel to it, cutting the circle at P and Q. Explain why sin POA = sin QOA = $\frac{2}{3}$. Measure these angles, and test your results by finding from the table that ·6665 = sin 41°48′.

20. By further measurement on the figure of Ex. 19 find the acute angles whose sines are $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$. Also find from the table the angles whose sines are nearest to ·2500, ·5000, ·7500.

21. Find from the table the angles whose sines are ·2639, ·7944, ·9720. Give an obtuse angle as well as an acute angle in each case.

22. Taking the nearest angle given in the table, find acute angles whose sines are ·55, ·7, ·8 and obtuse angles whose sines are ·6 and ·9. [Take ·7 as ·7000, and so on.]

23. Find from the table the angle whose sine is twice sin 21°.

[Note that this is not 42° but is larger.]

Using Sines to Find Angles [Draw a rough figure in each case.]

24. If, in Fig. 5, PN is 2,620 ft., find ∠PON.

[Sol'n. $\sin PON = 2620/3000 \bumpeq \cdot 8733$

$\therefore \angle PON \bumpeq 60°48'$ or $60\cdot 8°$. ($60°\ 51'$ is nearer).]

25. A ladder 18 ft. long rests against a wall at a height 16 ft. from the ground. Find the angle between the ladder and the ground.

26. A balloon at a height of 2500 ft. is connected with the ground by a cable 3600 ft. long. If the cable is assumed to be straight, what angle does it make with the ground ?

27. A trapdoor, horizontal when closed, is raised so that its edge which is 5 ft. from the hinge is 2 ft. 4 in. vertically above its closed position. What angle does the trapdoor make with the horizontal ?

28. A room is 10 ft. high. A cord stretched from one top corner to the opposite bottom corner is 32 ft. long. What angle does this make with the horizontal ?

29. In the parallelogram ABCD the perpendicular distance between AD and BC is 5 cm., while the length of AB is 6 cm. Find ∠ABC.

30. P is a point on the semi-circle when diameter is AB. If AB = 8 in.. ∠ABP = 62°, find the length of AP. Find also the length of BP.

[It is easier to use ∠PAB, than to use Pythagoras' theorem].

31. An isosceles triangle has each of the equal sides 9 in. and the vertex is at a distance 8 in. from the middle point of the base. Show that each of the equal angles is approximately 62·7°.

The Right-angled Triangle Definition

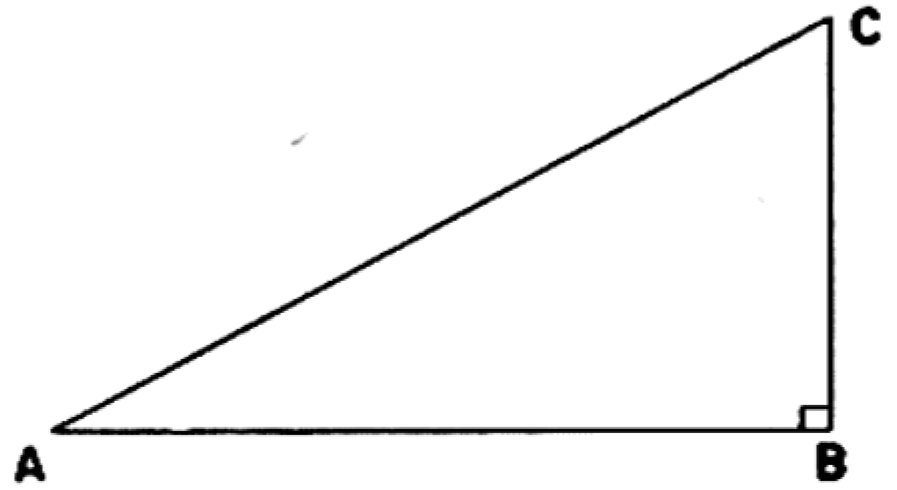

FIG 6.

Compare Fig. 6 with Fig. 2, p. 2.

You will see that, since ∠B is a right angle,

$$\sin A = \frac{BC}{AC} = \frac{\text{side opposite } \angle A}{\text{hypotenuse}}$$

or, for short, $\sin A = \frac{\text{opp.}}{\text{hyp.}}$

For an acute angle this may be taken as the definition of the sine.

[O.H.M.S. may help in remembering this : Opposite over Hypotenuse Means Sine.]

N.B. This is only true for an acute angle. You will see that in Fig. 3, p. 2, the height PN is not opposite to the obtuse angle AOP.

Examples—2

1. In Fig. 6, which is the side opposite the angle C ? Use the definition opp./hyp. to decide which ratio of sides is sin C.

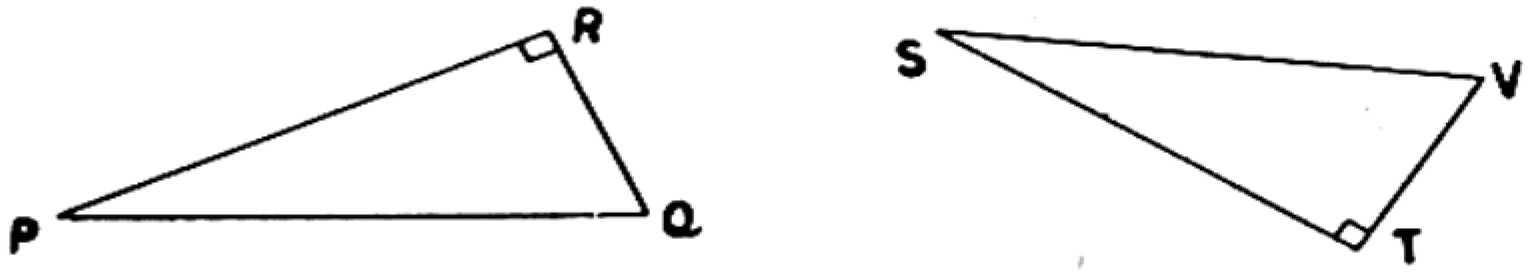

FIG. 7.

2. In Fig. 7, R and T are right angles. Write down as ratios of sides (i) sin P and sin Q, (ii) sin S and sin V.

3. Draw the triangle whose sides are 3 in., 4 in., 5 in. You know (or will see) that it is right-angled. Write down the numerical values of the sines of the two acute angles. Use the table to find the angles, and check by measurement on an accurate figure.

4. In Fig. 7, if PR $= 15$ cm., RQ $= 8$ cm., use Pythagoras' Theorem to prove that PQ $= 17$ cm.; hence write down sin P and find $\angle$P from the table.

[Sol'n. $PR^2 = 15^2 = 225$, $RQ^2 = 8^2 = 64$, $PR^2 + RQ^2 = 289$.
But $PQ^2 = PR^2 + RQ^2 = 289 = 17^2$ $\quad \therefore PQ = 17$.

$$\therefore \sin P = \frac{\text{opp.}}{\text{hyp.}} = \frac{8}{17} \simeq \cdot 4706 \qquad \begin{array}{l} 17)8{\cdot}00(\cdot 4706 \\ \underline{\;6{\cdot}8\;\;} \\ \;1{\cdot}20 \\ \underline{\;1{\cdot}19\;\;} \\ \quad 100 \\ \quad \underline{102} \end{array}$$

$$\therefore P \simeq 28°6'.]$$

5. In Fig. 7, ST $= 12$ cm., VT $= 5$ cm.; use Pythagoras' Theorem to show that SV $= 13$ cm. Hence write down sin S and sin V as fractions. Turn sin S into a decimal and find $\angle$S from tables.

6. In Fig. 7, if ST $= 15$ cm. and $\angle$VST $= 21{\cdot}2°$, find the length of the perpendicular from T to SV.

7. In $\triangle$ABC, AB $= 20$ cm., BC $= 25$ cm., CA $= 15$ cm. As this is the 3 : 4 : 5 triangle, enlarged, $\angle$A is a right angle. Write down sin B and sin C. Draw AN the perpendicular from A to BC. Find AN from $\triangle$ABN as AB sin B and also find it from $\triangle$ACN, showing that the two results are the same.

Verify also that AN . BC $=$ AB . AC. (Note that each is twice area of $\triangle$ABC.)

The Isosceles Triangle

The isosceles triangle *must be divided into two right-angled triangles* before the table of sines is used.

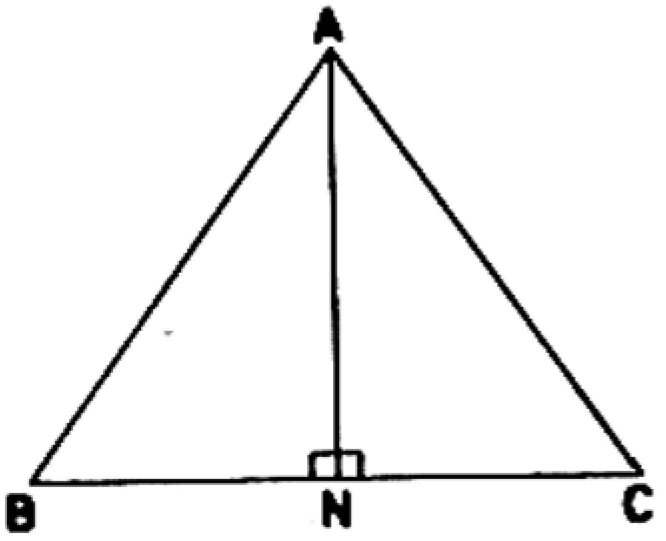

FIG. 8.

8. In Fig. 8, AB $=$ AC $= 9$ in., BC $= 6$ in.; find $\angle$BAC.

[Sol'n. Draw AN to bisect $\angle$BAC and hence to bisect BC at right angles.

BN $= \frac{1}{2}$BC $= 3$ in.; sin BAN $=$ BN/AB $= \frac{3}{9} = \cdot 3333$.

$\therefore \angle$BAN $= 19°30'$ and $\angle$BAC $\simeq 39°$.

Using the difference columns in four-figure tables,

$$\angle BAN = 19°28', \ \angle BAC = 38°56'.]$$

9. The equal sides of an isosceles triangle are each 10 in. Find the angle between them (i) if the base is 16 in., (ii) if the base is 9 in.

10. The equal sides of an isosceles triangle are each 25 cm. Find the base if the angle between the equal sides is (i) 52·6°, (ii) 147°.

11. The arms of a pair of dividers are both $5\frac{1}{2}$ in. long. If they are used to measure a line 7 in. long, what is the angle between them ?

12. One triangle has sides 6 cm., 8 cm., 8 cm., and another has sides 12 cm., 8 cm., 8 cm. Show that the angle between the equal sides in the second triangle is more than double that in the first triangle, by calculating both angles.

Use of the Complementary Angle

N.B. Two angles are complementary if their sum is 90°.

13. The base angles of an isosceles triangle are both 69°. Find the ratio of the base to one of the equal sides.

[Sol'n. In Fig. 8, $\angle ABN = 69^\circ$,

$$\therefore\ \angle BAN = 21^\circ\ (90^\circ - 69^\circ = 21^\circ).$$

$$\therefore\ BN/AB = \sin 21^\circ = {\cdot}3584. \quad \text{But } BC = 2BN,$$

$$\therefore\ BC/AB = {\cdot}3584 \times 2 = {\cdot}7168.]$$

14. Repeat Ex. 13 when each base angle is 35°.

15. A 12-ft. ladder rests against a wall and makes 72° with the ground. How far is its foot from the wall ? [Use sin 18°.]

16. I walk a mile in a direction 50° East of North. How far have I gone northward ?

17. A weight fastened to a string 7 ft. long swings out to $18\frac{1}{2}^\circ$ with the vertical. How far is it sideways from the point of support and how much is it below that point ?

Area of Parallelogram and Triangle

18. A parallelogram ABCD has $AB = 9$ in., $BC = 7$ in., $\angle ABC = 64^\circ$. Find its area.

[Sol'n. Copy Fig. 9, with AN perp. to BC. Then

$AN/AB = \sin 64^\circ = {\cdot}8988$

$\therefore\ AN = 9 \times {\cdot}8988 = 0{\cdot}0892$

$\therefore\ \text{Area} = AN\,.\,B\ \ C = 8{\cdot}0892 \times 7$

$= 56{\cdot}6244 \simeq 56{\cdot}6$ sq. in.]

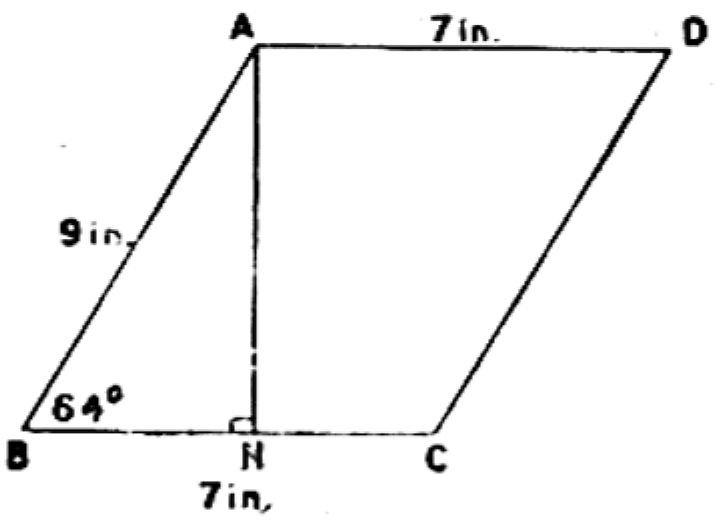

FIG. 9.

19. Find the areas of parallelograms in which
(i) sides 12 cm. and 15 cm. contain an angle 80°.
(ii) ,, ,, ,, ,, ,, ,, 100°.
(iii) sides 14 in. and 11 in. contain an angle 17°.

20. In Fig. 9, find the area of $\triangle ABC$; find also the areas of triangles in which (i) sides 8 in. and 5 in. contain 48°.
(ii) sides 14 cm. and 9 cm. contain 124°.

21. Give the general formulæ for the areas of
(i) a parallelogram in which sides a in. and b in. contain angle θ.
(ii) a triangle with the same sides and contained angle.

The Circle : Chords and Tangents

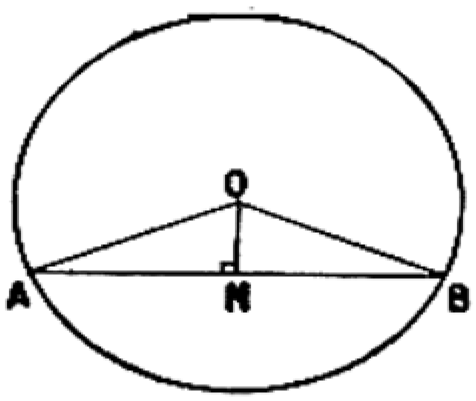

FIG. 10.

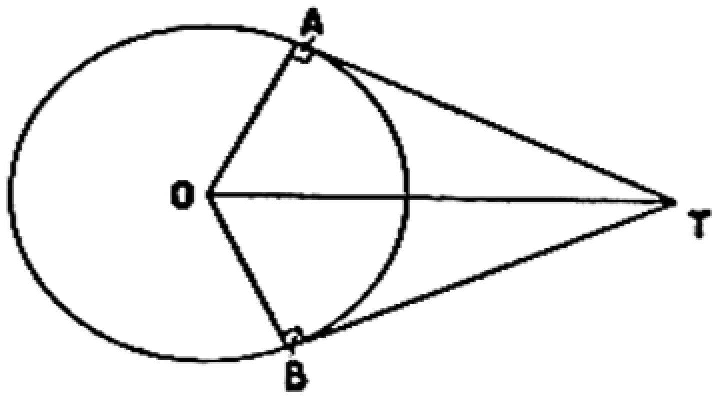

FIG. 11.

22. In Fig. 10 the radius AO = 7 cm. and $\angle AON = 60°$; find AB.

23. (i) The chord AB of a circle subtends 106° at the centre. If the radius is 25 in. find the length of AB.

[In Fig. 10, AB subtends $\angle AOB$.]

(ii) What angle does a chord 16 in. long subtend at the centre of a circle whose radius is 10 in. ?

24. In Fig. 11 the distance of T from the centre O is 14 cm. and $\angle AOT$ is 57°. Find the length of the tangents TA, TB (which are perpendicular to OA, OB). Find also the angle between the tangents, and the radius of the circle. [Use $\angle OTA$.]

25. Tangents are drawn to a circle of radius 4 in. from a point 8 in. from the centre. Show that the tangents contain 60°. What would be the angle between tangents to the same circle drawn from a point 10 in. from the centre ?

26. Find the length of the side of a regular pentagon (5-sided figure) inscribed in a circle of radius 10 cm.

27. With the data of Ex. 24 find (i) the length of AB, (ii) the lengths of the parts into which AB divides OT.

28. Radii OA, OB of a circle make 70° with each other and are 8 in. long. (i) Find area of $\triangle$OAB from the formula $\frac{1}{2}$OA . OB sin 70° [Ex. 21 (ii)]. (ii) Drawing ON perp. to AB use $\angle$AON $= 35°$ to find AN, and $\angle$OAN $= 55°$ to find ON. Hence find area of $\triangle$OAB.

Examples—3. Miscellaneous

1. Show that the triangle whose sides are 7 in., 24 in., 25 in. is right-angled. What are the sines of its acute angles? Find these angles.

2. OA is the radius of a circle centre O, and OA $= 3$ in. AB is a chord such that $\angle$OAB $= 41{\cdot}2°$. Find the perpendicular distance from O to AB.

3. The slant height of a cone is 7 cm. and the vertical height 5 cm. Find the angle which the slant height makes with the base. Find also the vertical angle, and the radius of the base.

4. O is the centre of a circle; AT is a tangent from an external point A. If OA is 9 in. and $\angle$TOA is $31{\cdot}1°$, find AT and OT.

5. An aeroplane rises at 4° to the horizontal. If its speed is 120 m.p.h. what is its altitude in feet after 2 minutes?

6. In a "lean-to" shed the width of the sloping roof is 8 ft. and one edge of this roof is 15 in. higher than the opposite edge. What is the angle of slope of the roof?

7. (i) If $\sin x = 3 \sin 15°$, by how much is x greater than 45°?

(ii) Why is there no angle θ such that $\sin \theta = 2 \sin 45°$?

8. Two beams BA,BC fastened together at B rest on the ground at A,C and carry a weight hung from B so that its lowest point P is 7 ft. below B. If the plane ABC is vertical and AB $=$ CB $= 17$ ft., AC $= 11$ ft., how high is P above the ground?

9. A lighthouse rises 40 ft. from the water and is opposite a vertical cliff. From the top of the cliff a rope 150 ft. long would reach either straight down to the foot of the cliff or straight to the top of the lighthouse. What angle will the rope make with the horizontal in the latter position?

Graph of sin θ

A convenient way of drawing the graph of sin θ is shown below. Angles are marked off round the circle and heights transferred horizontally as shown.

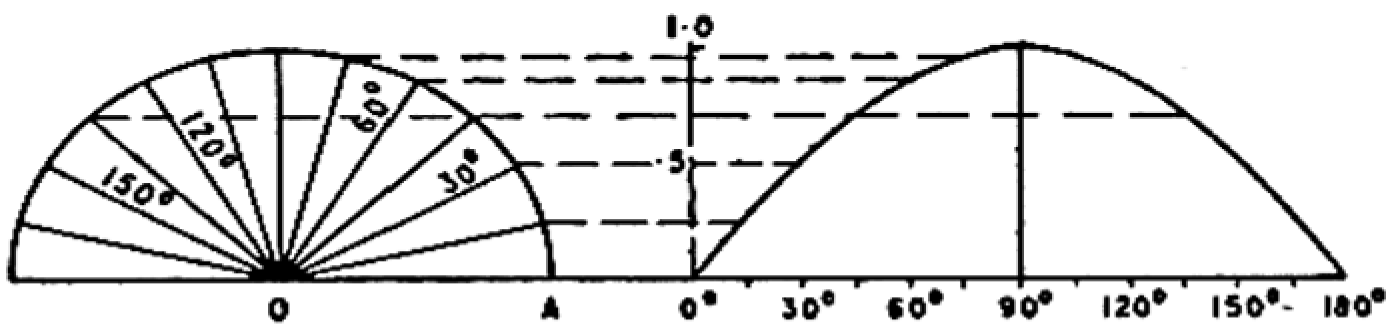

FIG. 12.

This method is independent of the tables, but the graph can also be drawn by using the tables.

Examples—4

1. Explain how the part of the graph from 90° to 180° is got from the part from 0° to 90°.

2. How does the graph show that $\sin(90° + A) = \sin(90° - A)$? How does this result follow from the definition of sine?

3. For what ranges of values of x
   (i) is $\sin x$ less than ·8?
   (ii) is $\sin x$ greater than ·2?

4. Use the graph to find two values of x from the equation $\sin x = {\cdot}6$.

5. Find roughly from the graph the angle θ if $\sin\theta = \frac{1}{2}\sin 60°$. Check by the use of tables.

6. Draw on the same axes the graphs of $\frac{1}{2}\sin x$, $\sin x$, $1\frac{1}{2}\sin x$ and $2\sin x$ from 0° to 180°, using tables.

7. Draw the graph of $\sin x$ from 0° to 180°, using 2 in. as unit for $\sin x$ and 4 in. for 180 degrees.

Draw on the same axes the parabola $y = 2x - \frac{1}{2}x^2$ (taking 1 in. as unit for both x and y) which has the same height and base as the sine graph.

[This shows that the sine curve is not a parabola.]

REVISION EXERCISES

IA

1. A ladder 10 ft. long leans against a wall with which it makes an angle of 37°. How far from the wall is the foot of the ladder ?

2. If, in the previous question, the foot of the ladder is moved so as to be half as far from the wall as before, what angle does the ladder now make with the wall ?

3. The sides of a rhombus are all 5 in. long, and its acute angles are 50°. Calculate the lengths of its diagonals.

[The diagonals of a rhombus cut at right-angles.]

4. Find the perpendicular distance between two opposite sides of the rhombus in the previous question.

IB

1. Find two values of the angle A if $\sin A = 1\frac{1}{2} \sin 40°$.

2. Show that the triangle whose sides are 20 in., 21 in. and 29 in. is right angled, and find the two acute angles.

3. The sloping sides of a railway cutting are 60 ft. long and make 33° with the horizontal (Fig. 13). The bottom of the cutting is 50 ft. wide. How deep is the cutting ?

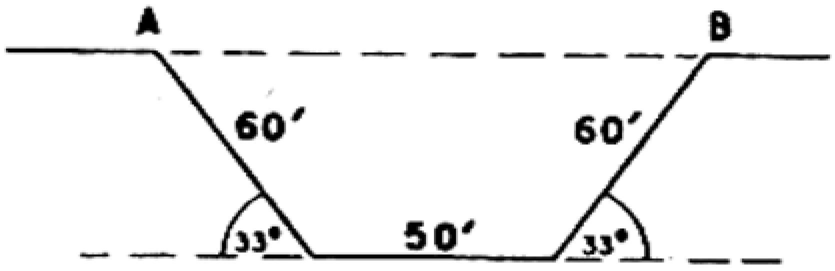

FIG. 13.

4. How wide is the cutting of Ex. 3 across AB ?

[Use the complementary angle.]

IC

1. A rope 15 ft. long hangs from the middle of the ceiling of a gymnasium 20 ft. wide. Through what angle can the rope swing on either side of the vertical if its end swings from one side wall to the other ?

2. A pole is inclined so that its shadow is as long as possible. The pole is then at right angles to the sun's rays. If the shadow is 10 ft. long when the rays make 49° with the horizontal, find the length of the pole.

3. The sides of a parallelogram are 4 in. and 5 in., and its area is 11 sq. in. Calculate (i) the distance between the longer sides and (ii) the acute angles of the parallelogram.

4. A cubical box whose edge is 5 ft. is tilted so that its base makes 30° with the ground (Fig. 14). Calculate the heights of B, C and D above the ground.

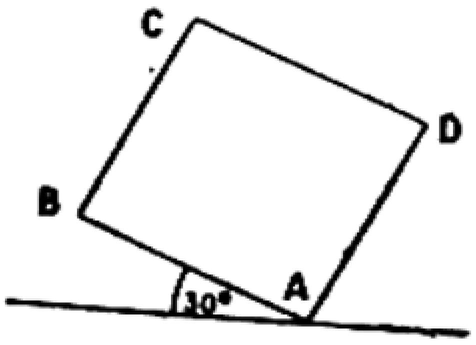

FIG. 14.

ID

1. ABC is a triangle with AB = AC = 10 in. Calculate BC (i) when $\angle A = 50°$, (ii) when $\angle B = 50°$.

2. A slate roof should make at least $25\frac{1}{2}°$ with the horizontal, according to a book on building. One such roof is 21 ft. from ridge to eaves, and the ridge is 9 ft. higher than the eaves. Is its slope great enough ?

3. A heap of dry sand of a certain type usually settles with its sides making 38° with the horizontal. The length of the sloping side of a heap is 8 ft. How high is the heap ?

4. There are two straight paths to the top of a hill. One slopes at 20° to the horizontal, the other at 15° and is 100 ft. longer. (i) How long is the shorter path ? (ii) How high is the hill ?

[If the shorter path is x ft. long, show that the height is $x \sin 20°$ and also $(x + 100) \sin 15°$. Solve this equation.]

IE

1. A triangle ABC is right angled at B, and the sides AB, AC are 2 ft. 11 in. and 3 ft. 1 in. respectively. Calculate the angle ACB and the perpendicular distance from B to AC.

2. An aircraft flies 15° off its course owing to a faulty compass. If its speed is 200 miles per hour, how far is it off its course after half an hour ? How far is it from the spot where it would have been if the compass had been correct ?

3. Find the area of a dodecagon the same size as the new nickel-brass threepenny pieces, which just fit into a circle of diameter $\frac{7}{8}$ in. [If O is the centre of the circle and AB a side of the dodecagon, find the perpendicular from A to OB and hence the area of $\triangle$AOB. Work in fractions.]

4. A destroyer D is due E. of a cruiser C which is steaming due N. at 20 knots. In what direction must the destroyer steam at 35 knots in order to intercept the cruiser ? [If they meet at M after x hours, what is the ratio of MC to MD ?]

CHAPTER II

THE SINE FORMULA

THE result to be proved and used in this chapter is that the *sides of a triangle are proportional to the sines of the opposite angles*.

This result, which is the most important in trigonometry, is called *the sine formula*, or very often *the sine-rule*.

In the usual notation in which side a is opposite angle A it is

$$\frac{\mathbf{a}}{\sin \mathbf{A}} = \frac{\mathbf{b}}{\sin \mathbf{B}} = \frac{\mathbf{c}}{\sin \mathbf{C}}$$

Proof of Sine Formula

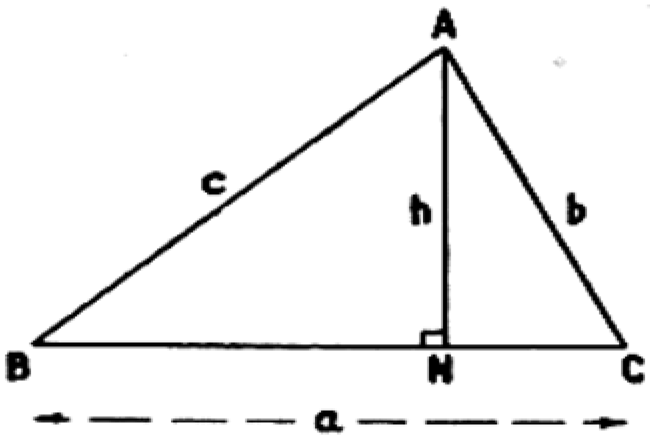

FIG. 15.

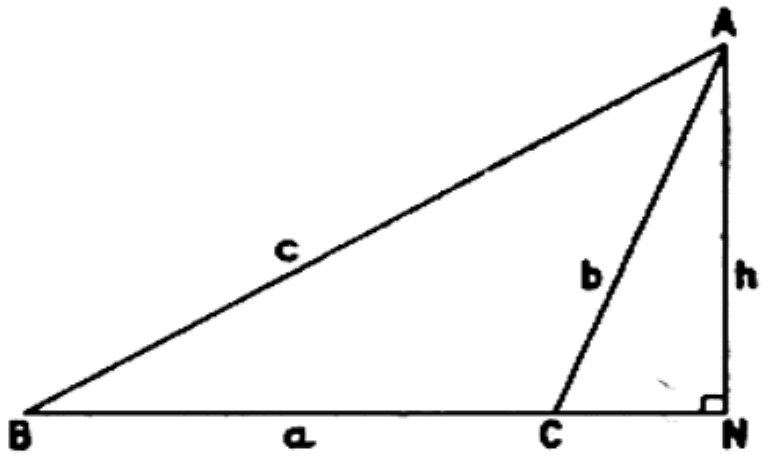

FIG. 16.

In Fig. 15, ABC is acute angled at B and C; AN is drawn perp. to BC.

In Fig. 16, ABC is obtuse angled at C and AN the perp. from A to BC meets BC produced at N. Let AN $= h$ in each figure.

In each figure two right-angled triangles ABN, ACN are formed.

In Fig. 15, from $\triangle$ABN, $\sin B = \frac{h}{c}$, from $\triangle$ACN, $\sin C = \frac{h}{b}$

$$\therefore\ c \sin B = b \sin C \text{ for each } = h.$$

$$\therefore\ \frac{\sin B}{b} = \frac{\sin C}{c}, \text{ or } \frac{b}{\sin B} = \frac{c}{\sin C}.$$

It follows from symmetry (or can be shown by drawing a perpendicular from one of the other angles) that each of these last $= \dfrac{a}{\sin A}$.

In Fig. 16 precisely the same argument applies, for the sine of the obtuse angle C $= \sin (180 - C) = \sin ACN = h/b$, so that $h/b = \sin C$.

Hence in both figures $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Memorise this formula. It is often used in the form

$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

Examples—5

1. (i.) In Fig. 15, by drawing a perp. from B to AC, prove that $c \sin A = a \sin C$.

(ii) In Fig. 16, what result is obtained by drawing a perp. from C to AB ?

2. The angles of a triangle are 64°, 48°, 68° ; find the ratio of the largest side to the smallest, (i) as a ratio of sines, (ii) as a decimal to two places.

[Sol'n. (i) By the sine formula the ratio is $\sin 68° : \sin 48°$.
(ii) As a decimal the ratio is $0{\cdot}9272 \div 0{\cdot}7431 \simeq 1{\cdot}25$.]

$$\begin{array}{r} 7431)9272(1{\cdot}25 \\ \underline{7431} \\ 18410 \\ \underline{14862} \\ 3548 \end{array}$$

3. Repeat Ex. 2 for the triangles whose angles are 37·3°, 70°, 72·7°. (ii) 30°, 40°, 110°. (iii) 53°, 42°30′, 84°30′.

4. Two angles of a triangle are 23°36′ and 45°6′ ; if the side opposite the former is 10 in. what is the side opposite the latter ?

5. What is the length of the side opposite an angle of 98° in a triangle in which a side 12·4 ft. is opposite an angle of 64°12′.

6. One side of a triangle is one-third of a second side ; if the angle opposite the larger is 138°, what is the angle opposite the smaller ?

[Sol'n. Sin 138° = sin 42° = 0·6691 ∴ the sine of the required angle = $\frac{1}{3}$ of 0·6691 = 0·2230 ∴ the angle is 12°54′. (12°53′).]

7. One side of a triangle is double another; find the angle opposite to the larger side, if that opposite the smaller is (i) 29°, (ii) 30°.

8. The sines of two angles of a triangle are 0·5 and 0·7. Find these angles and the third angle. Find also the ratio of each of the other sides to the smallest side.

The Sine Formula is also true for a right-angled triangle, as can be seen by working Ex. 9, 10. *Remember that sin* 90° = 1.

9. In Fig. 17, sin A = a/b; write down a similar expression for sin C and deduce that

$$\frac{\sin A}{\sin C} = \frac{c}{a}.$$

10. In Fig. 17, since sin B = 1, $\frac{\sin A}{\sin B} = \frac{\sin A}{1} = \frac{a}{b}$, give the corresponding work to prove that $\frac{\sin C}{\sin B} = \frac{c}{b}$.

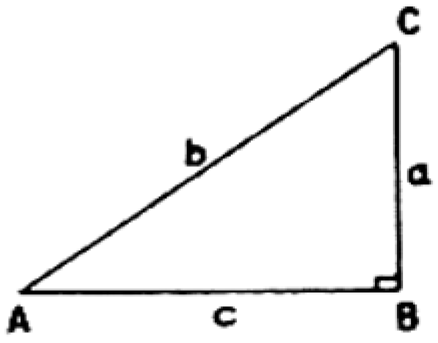

FIG. 17.

[Ex. 11, 12, 13, 14 refer to Fig. 17.]

11. If a = 8 in., b = 17 in., c = 15 in., which makes ∠B = 90°, write down the fractions equal to sin A and sin C and verify that sin A/sin C = a/c.

12. If ∠A = 32° find the ratio $a : c$.

[Sol'n. If A = 32° then C = 58° to make up 90°.

$$\therefore \frac{a}{c} = \frac{\sin A}{\sin C} = \frac{\sin 32^\circ}{\sin 58^\circ} = \frac{0{\cdot}5299}{0{\cdot}8480}$$

∴ $a : c$ = 0·625

848)5299(0·625
5088
2110
1696
4140]

13. If C = 54°30′, find the ratio $a : c$.

14. If A = 27°6′, find the ratios $a : b : c$.

15. In a triangle two angles are 36·9° and 53·1° and the smallest side is 18 in. Find the side opposite 53·1°.

16. In an isosceles triangle whose base angles are each 70°, find the ratio of the altitude to the base, first in terms of sines and then as a decimal. (Remember to divide the triangle into two right angled triangles.)

17. If the vertical angle of an isosceles triangle is 132°48′, prove that the base is just over four and a half times the altitude.

Angles of Elevation and Depression

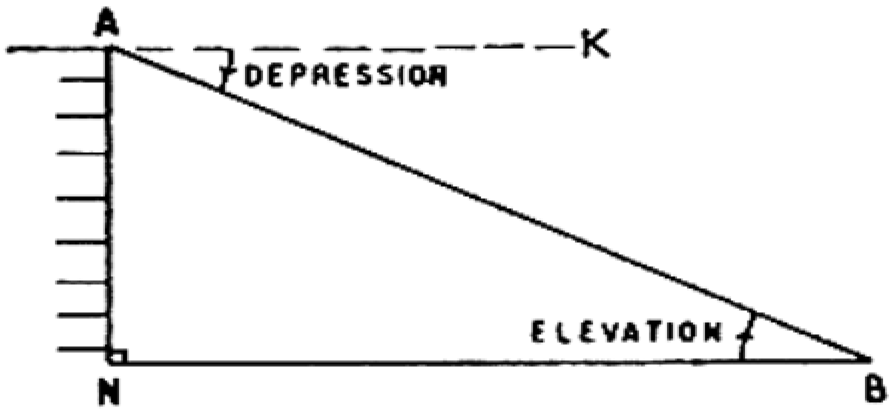

FIG. 18.

Suppose A the top of a cliff and B a boat. One can look up from B to A and see A at an *angle of elevation* ABN, or look down from A to B and see B at an *angle of depression* BAK.

The angle of depression is measured from the *horizontal* downwards, the angle of elevation is measured from the *horizontal* upwards. Of course the two angles marked are equal, being alternate angles.

Examples—6

These examples are worked much more easily by the use of the *tangent* ratio explained in Chapter IV.

1. In Fig. 18, if the angle of depression of B from A is 13°18′ and the cliff AN is 115 ft. high, find the distance NB.

[Sol'n. $\angle NAB = 90° - 13°18' = 76°42'$.

$$\frac{NB}{AN} = \frac{\sin NAB}{\sin ABN} = \frac{\sin 76°42'}{\sin 13°18'} = \frac{0{\cdot}9732}{0{\cdot}2300}$$

$$NB \simeq 115 \times 97{\cdot}3 \div 23 \text{ ft.} = 456{\cdot}5 \text{ ft.}]$$

2. Repeat Ex. 1, taking the height of the cliff to be 124 ft. and the angle of depression of the boat from its top to be 38°18′.

3. From a point 200 ft. from the foot of the tower, and on the same level, the angle of elevation of the top of a tower is 45·6°. Find the height of the tower.

4. How tall is a tree if it casts a shadow 90 ft. long when the altitude of the sun is 35·9°. (The altitude of the sun is its angle of elevation.)

5. Find the elevation of the top T of a tree 45 ft. high from a point P at a distance of 70 ft. from the foot F.

[Calculate TP by " Pythagoras " and then find sin TPF.]

Chord and Angle at Circumference : Radius of Circumcircle

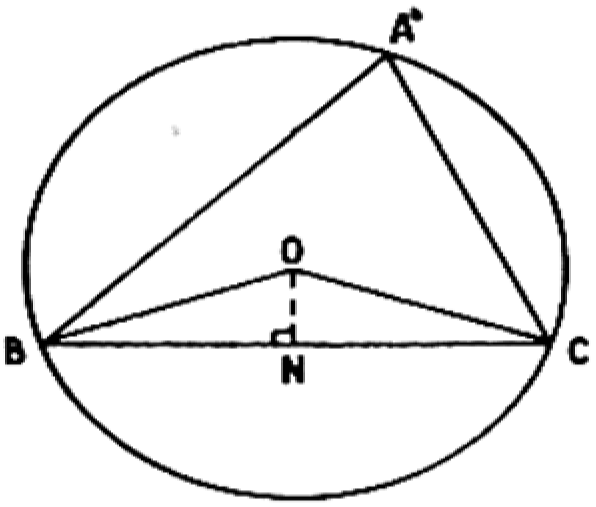

FIG. 19.

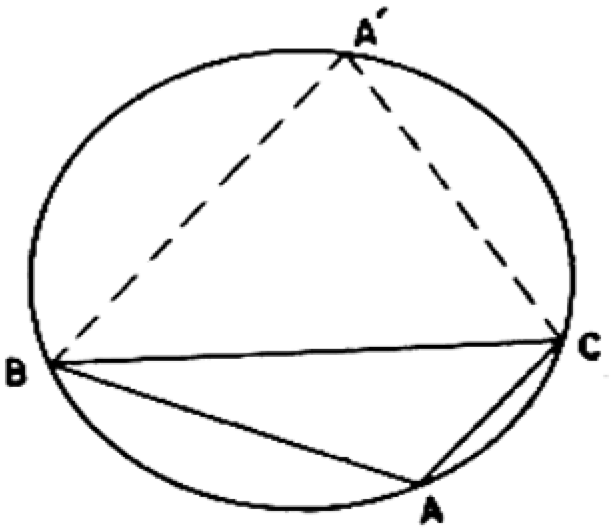

FIG. 20.

In Fig. 19 a chord BC subtends an acute angle A at the circumference of a circle, radius R. ON is perp. to BC.

$\therefore\ \angle BOC = 2A$ and $\angle BON = A$ (being half of $\angle BOC$).

$$\therefore\ BC = 2BN = 2R \sin A.$$

In Fig. 20, where $\angle A$ is obtuse, $\angle BA'C$ is acute since $\angle BA'C = 180° - A$.

Now $BC = 2R \sin A'$, where A′ is any point on the larger arc BC.

But $A' = 180° - A$, so that $\sin A' = \sin A$.

$\therefore\ BC = 2R \sin A$ as in the case when $\angle A$ is acute.

Similar results are obtained for the other sides :

$$AC = 2R \sin B,\ AB = 2R \sin C.$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

This is an extended version of the sine formula.

Examples—7

1. In a right-angled triangle why is the hypotenuse equal to 2R ? Verify the formula just given for the 3, 4, 5 triangle.

2. In a circle of radius 8 cm. a triangle is inscribed, with two angles 50° and 70°. Find the sides of the triangle.

[Sol'n. The side opposite $50° = 2R \sin 50° = 16 \times \cdot7660 = 12\cdot26$, etc.]

3. A chord 14 in. long subtends 20°30′ at the circumference of a circle; find the diameter of the circle. [You must *divide* by ·35 or ·3502.]

4. In a circle of diameter 10 cm. a chord 9 cm. long is placed. What angle does it subtend at the circumference ?

5. Which is the larger chord of a circle, one that subtends 120° at the centre or one that subtends 120° at the circumference ?

Solution of Triangles by Sine Formula

The various cases of congruence of triangles, denoted by S.A.S., S.S.S., etc., indicate the different selections of three parts (sides or angles) in a triangle from which the other parts may be found.

Two of these cases depend directly on the sine formula.

In general, logarithms should be used in solving triangles, but the examples given in this page and the next are fairly simple ones and can be done without logarithms.

Examples—8

Case A.A.S.

1. In $\triangle ABC$, $\sin A = \cdot6$, $\sin C = \cdot7$; if $a = 30$ yards, find c.

2. In $\triangle ABC$, $A = 30°$, $\sin B = \cdot8$; if $b = 44$ ft., find a.

3. In $\triangle PQR$, $P = 30°$, $Q = 36°54'$; if $p = 17$ in., find q.

4. In $\triangle ABC$, $B = 150°$, $b = 9$ ft.; if $\sin A = \cdot3$, prove that $a = 5\cdot4$ ft.

5. In $\triangle ABC$, $\angle C = 122°$, $a = 10$ in.; if $B = 28°$ find c (first find A).

Case A.S.S. (The ambiguous case.)

Examples 6 to 10 should be checked by accurate drawing.

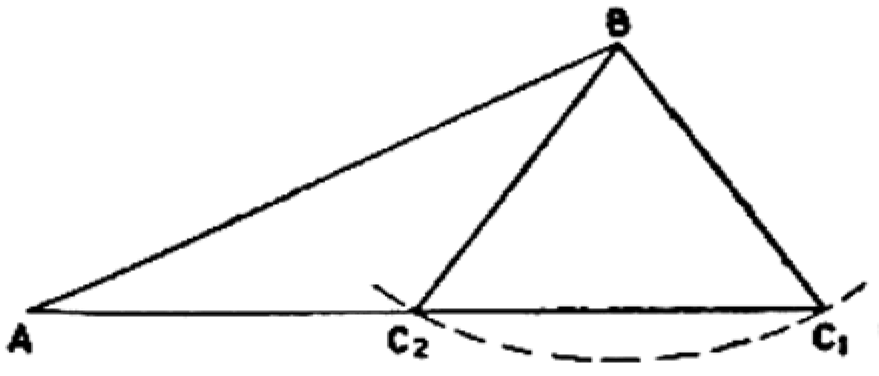

FIG. 21.

6. In $\triangle ABC$ given $a = 4$ in., $c = 7$ in. and $A = 27°$, find sin C. Show that C may be either acute or obtuse so that there are two possible triangles (ABC_1 and ABC_2 in Fig. 21). Find the third angle in each case.

[Sol'n. $\dfrac{\sin C}{7} = \dfrac{\sin A}{4} = \dfrac{\cdot4540}{4}$ $\therefore$ $\sin C = \dfrac{3\cdot178}{4} = \cdot7945.$

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$\cdot 7945 \simeq \sin 52^\circ 36'$, which is also $\sin 127^\circ 24'$, since these two angles make up 180°.

$\therefore\ \angle C$ may be $52^\circ 36'$ ($\triangle ABC_1$) which makes $\angle B = 100^\circ 24'$ ($\angle ABC_1$) or $\angle C$ may be $127^\circ 24'$ ($\triangle ABC_2$) which makes $\angle B = 25\cdot 36'$ ($\angle ABC_2$).]

7. In $\triangle ABC$ given $a = 5$ cm., $c = 6$ cm. and $\angle A = 40^\circ 24'$, find sin C and two possible values for C; find also the third angle in each case.

8. Repeat Ex. 7, given $a = 8$ cm., $c = 9$ cm. and $A = 20\cdot 6^\circ$.

9. (i) In $\triangle ABC$, if $a = 4$ in., $c = 7$ in. and $A = 54^\circ$, use the sine formula to find sin C. Why is the result impossible? Illustrate by a figure.

(ii) In $\triangle ABC$, if $a = 7$ in., $c = 4$ in., and $A = 54^\circ$, find sin C and show that the obtuse value for C is impossible since it makes $A + C > 180^\circ$.

Find the acute value for C and the third angle in this case.

10. Find the other angles of the triangles (if they can be drawn) for which

(i) $a = 6$ cm., $c = 5$ cm. and $A = 40\cdot 4^\circ$.
(ii) $a = 3$ cm., $c = 6$ cm. and $A = 40\cdot 4^\circ$.
(iii) $b = 5$ cm., $a = 11$ cm. and $B = 25\cdot 4^\circ$.

When the angles are known the remaining side can be found by the sine formula.

[Thus in Ex. 6

$$\frac{AC_2}{\sin 25^\circ 36'} = \frac{4}{\sin 27^\circ} \qquad AC_2 = \frac{4 \times \cdot 4321}{\cdot 4540} = 3\cdot 807 \text{ in.}$$

$$\begin{array}{r} \cdot 454)1\cdot 7284(3\cdot 807 \\ \underline{1\cdot 362} \\ 3664 \\ \underline{3632} \\ 3200.] \end{array}$$

11. In Ex. 7 find the third side of the obtuse-angled triangle.

12. In Ex. 8 find the third side of the triangle which has the smallest angle.

Repeated use of Sine Formula

It is an important consequence of the sine-rule that, *if a figure is made up of triangles and all the angles are known, and if one side is given, then any other side can be found.*

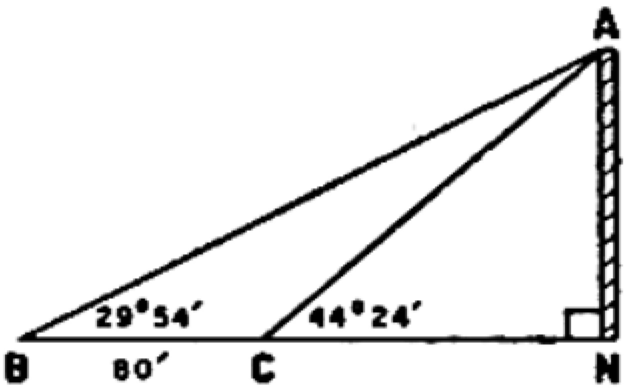

FIG. 22.

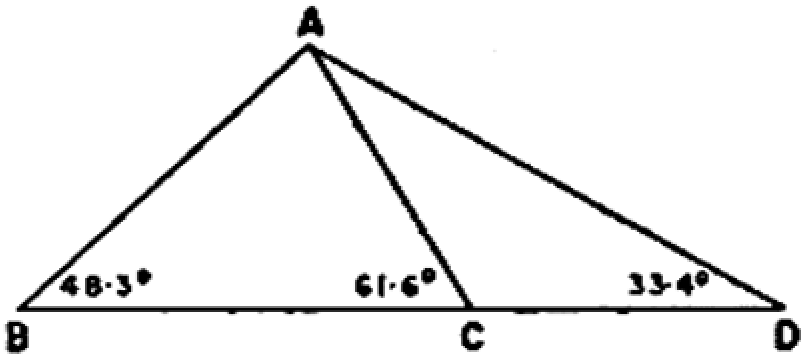

FIG 23.

Examples—9

1. In Fig. 22 a tower AN is observed from two points B, C, 80 ft. apart and in a line with its foot on the same level. The angles of elevation are as shown in the figure. Find the height of the tower.

[Sol'n. $\angle BAC = 14°30'$ since ext. angle = sum of int. opp. angles.

By sine-rule $AB/80 = \sin 44°24'/\sin 14°30'$.

Also from $\triangle ABN$, $AN = AB \sin 29°54'$.

From these show that $AB = 80 \times \cdot6997 \times \cdot4985 \div \cdot2504$ and find AB approximately as $80 \times \cdot7 \times \cdot5 \div \cdot25$.]

2. In Fig. 23, A is an aeroplane observed from three points B, C, D in a straight line. If B, D are a mile apart, find AC in yards.

[Hint. Find AB/BD and AC/AB.]

3. If, in Fig. 23, ABCD is a vertical plane, find the *height* of the aeroplane above BD.

Use of Four-figure Tables

Four-figure tables of sines are very similar to tables of four-figure logarithms. You should follow through Ex. 1 and Ex. 2 (below) with a table of "natural sines" * open before you.

Examples—10

1. Find sin 9°44′ and sin 72°17′.

[Sol'n. $\sin 9°42' = \cdot1685$. For the extra 2′, look in the column headed 2′ at the right-hand side of the page, on the same row, and you find 6. Add ·0006 to ·1685; $\sin 9°44' = \cdot1691$.

---

* Natural sines are the ordinary sines; logarithmic sines are their logarithms.

Sin 72°12′ = ·9521 ; under 5′ you find 4, giving ·9521 + ·0004 = ·9525. It is, however, usually more accurate to work from the *nearest* entry in the main table ; sin 72°18′ = ·9527, and the difference for 1′ is ·0001. This makes sin 72°17′ = ·9527 — ·0001 = ·9526.

(Five-figure tables give sin 72°17′ = ·95257.)]

2. Find the angles whose sines are ·5319 and ·9733.

[Sol'n. The number next below ·5319 in the main table is ·5314, which is sin 32°6′. This leaves a difference ·0005, and 5 comes under 2′. ∴ ·5319 = sin 32°8′.

The number next below ·9733 is ·9732, which is sin 76°42′. This leaves ·0001, and 1 appears in the difference columns under both 1′ and 2′. So all that can be settled is that ·9733 is sin 76°43′ or sin 76°44′.

(Five-figure tables give ·97330 = sin 76°43·8′.)]

3. Find the sines of 24°39′, 32°26′, 57°46′, 80°20′, 86°25′.

[Sol'n for the last one. Sin 86°25′ is between ·9980 which is sin 86°24′ and ·9981 which is sin 86°30′. As 25′ is nearer 24′ than 30′, we take ·9980.]

4. Find the angles whose sines are : ·1467, ·4007, ·7499, ·9012, ·9835.

5. Show that, *as far as the four-figure tables tell*, the sines ·5406, ·5407, ·5408 all give the angle 32°44′, while all the angles from 87°48′ to 87°54′ have the same sine.

In longer calculations it is almost essential to use logarithms. To find the logarithm of, say, sin 48°, it is not necessary to look out sin 48° (which is ·7431) and then look out log ·7431 (which is $\bar{1}$·8711). The table of *logarithmic sines* gives $\bar{1}$·8711 directly. [Log. sin = log. of sine.]

There is difficulty with the difference columns up to about 3° in this table, and different books use different arrangements.

6. Find the log. sines of 76°54′, 33°44′, 11°33′, 5°48′, 6°40′.

7. Find the angles whose log. sines are $\bar{1}$·6600, $\bar{1}$·6789, $\bar{1}$·9753, $\bar{1}$·0120, $\bar{2}$·7939.

The following examples are intended to be done with four-figure tables and logarithms. They should be omitted unless logarithms can be used.

8. Two angles of a triangle are 48° and 68°. Find as a

decimal the ratio of the smaller to the larger of the sides opposite them.

[Sol'n. By the sine formula the ratio is

$$\frac{\sin 48^\circ}{\sin 68^\circ} = \cdot 8015 \qquad \begin{array}{r} \text{Logs} \\ \bar{1}\cdot 8711 \\ \underline{\bar{1}\cdot 9672} \\ 1\cdot 9039 \end{array}$$

Note that the logarithms are found *directly* from the table of " logarithmic sines."]

9. Two sides of a triangle are 4·2″ and 5·6″; the angle opposite the former is 31°48′; find the other angles.

[Sol'n. Call the given sides a and b, then $A = 31^\circ 48'$

$$\frac{\sin B}{\sin A} = \frac{b}{a} = \frac{56}{42} \quad \therefore \sin B = \frac{56 \sin 31^\circ 48'}{42} \qquad \begin{array}{r} \text{Logs} \\ 1\cdot 7482 \\ \underline{\bar{1}\cdot 7218} \\ 1\cdot 4700 \\ \underline{1\cdot 6232} \\ \bar{1}\cdot 8468] \end{array}$$

$\therefore B = 44^\circ 39'$ or $135^\circ 21'$ (its supplement)

$\therefore C = 103^\circ 33'$ or $12^\circ 51'$.

Examples 3, 4, 5 refer to △ABC with the usual notation.

10. $A = 73^\circ 14'$, $B = 51^\circ 12'$; find b as a decimal of a.

11. (i) If $b = 42\cdot 3$ yd., $B = 17^\circ 40'$, $C = 82^\circ 12'$, find c.
   (ii) If $A = 72^\circ 49'$, $B = 12^\circ 30'$, find b as a decimal of c.
   (iii) If $b = 8\cdot 48$, $c = 4\cdot 14$, $C = 30^\circ$, show that B cannot be found. Draw a figure.

12. (i) If $b = 3\cdot 78''$, $c = 4\cdot 53''$, $C = 48^\circ 49'$, find sin B and show that only one value of B is possible. Draw a figure.
   (ii) If $b = 7\cdot 36$, $c = 5\cdot 12$, $C = 39^\circ 4'$, find two possible values for B.
   (iii) If $b = 11\cdot 2$, $c = 6\cdot 72$, $C = 36^\circ 52'$, are there two values for B ?

13. B, C, D are three points in this order on a straight line and A is a point outside this line. If $\angle ABC = 38^\circ$, $\angle BAC = 31^\circ$, $\angle CAD = 43^\circ$, determine the ratio of BC to CD.

14. The beam of a searchlight starting from A makes 61° with the upward vertical. Another from B, which is 3 miles from A makes 34° with the upward vertical, both beams being in the vertical plane through AB. The beams meet at C. Find the length of BC. Hence find the distance from B of the point on AB vertically below C (which is between A and B).

Figures in Three Dimensions

Figs. 24, 25 show the observations which can be made to find the height of P above A when the base line AB cannot be taken in the same vertical plane as AP.

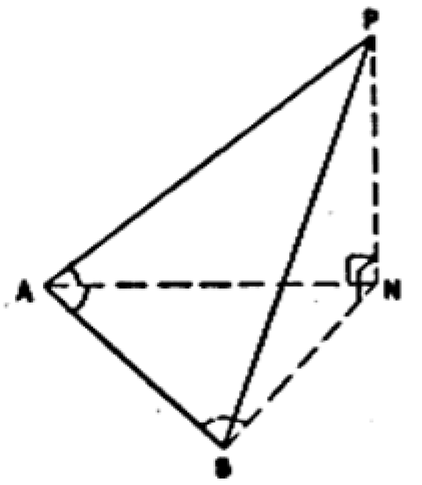

FIG. 24.

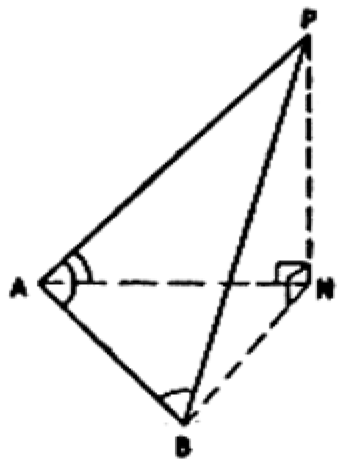

FIG. 25.

In Fig. 24 AB is horizontal and P is visible from both A and B, though N is probably not visible. A theodolite is used to measure the difference of bearing of P and A as seen from B. This is the angle between vertical planes through lines BA and BP and so is the angle NBA. Similarly the angle NAB is found by taking the bearings of B and P from A. The angle of elevation NAP is also measured and the length of the base line AB. Then AN : AB is found from △ANB and PN : AN from △PAN. This is the usual method of finding the height PN. It can be adapted to the case when AB is not horizontal.

In Fig. 25, ∠s PAB, PBA are measured with a sextant and also AB and ∠PAN as before. Then PA : AB is found from △PAB and PN : PA from △PAN.

In this method AB need not be horizontal.

Examples—11

(suitable for use of logarithms).

1. In Fig. 24, AB = 80 ft., ∠NAB = 46°, ∠NBA = 72°22′, ∠PAN = 41°19′; find the height of P above A, and also find AN.

[Sol'n. ∠ANB = 61°38′ (third angle of △) and ∠APN = 48°41′.

$$\therefore \frac{AN}{AB}=\frac{\sin 72^\circ 22'}{\sin 61^\circ 38'} \quad \text{and} \quad \frac{PN}{AN}=\frac{\sin 41^\circ 19'}{\sin 48^\circ 41'}$$

$$\therefore AN=\frac{80\sin 72^\circ 22'}{\sin 61^\circ 38'} \qquad \therefore PN=\frac{AN\sin 41^\circ 19'}{\sin 48^\circ 41'}$$

$$=86\cdot 66 \text{ ft.} \qquad\qquad =76\cdot 15 \text{ ft.}$$

Logs	Logs
1·9031	1·9378
$\bar{1}$·9791	$\bar{1}$·8196
1·8822	1·7574
$\bar{1}$·9444	$\bar{1}$·8757
1·9378	1·8817]

2. In Fig. 24, $AB = 240$ ft., $\angle NAB = 73^\circ 10'$, $\angle NBA = 54^\circ 30'$, $\angle PAN = 52^\circ 15'$, find AN and the height PN.

3. In Fig. 25, AB is 50 yds., $\angle PAB = 63^\circ 14'$, $\angle PBA = 71^\circ 12'$, $\angle PAN = 36^\circ 4'$, show that PN is about 118 ft.

4. In Fig. 25, $AB = 408$ ft., $\angle PAB = 73^\circ 10'$, $\angle PBA = 64^\circ 33'$, $\angle PAN = 38^\circ 16'$, find PN.

5. In Fig. 25, $\angle PAB = \angle PBA = 70^\circ$; find $\angle PAN$, if $PN = AB$.

REVISION EXERCISES

IIA

1. Two ships set out from the same port at the same time, and with the same average speed of 15 knots. Their tracks differ by 28°. How far apart are they (in sea miles) after four hours?

[A knot is a speed of one sea-mile per hour.]

2. Find the area of a triangle ABC whose sides AB, AC are 15 in. and 16 in. respectively and contain an angle of $39^\circ 8'$.

[Find the perpendicular distance from B to AC.]

3. The chord of an aeroplane wing-section is 10 ft., and the angle of attack is 12° (Fig. 26). If the height of the leading edge above the ground is 8 ft. 3 in., what is the height of the trailing edge?

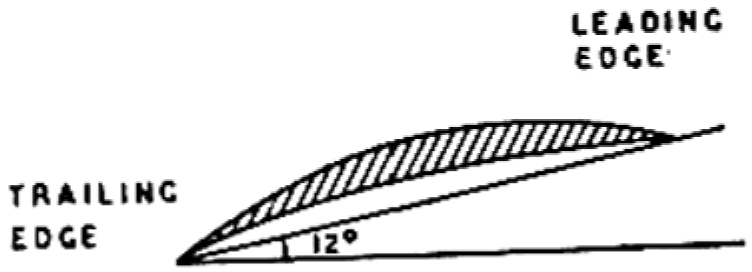

FIG. 26.

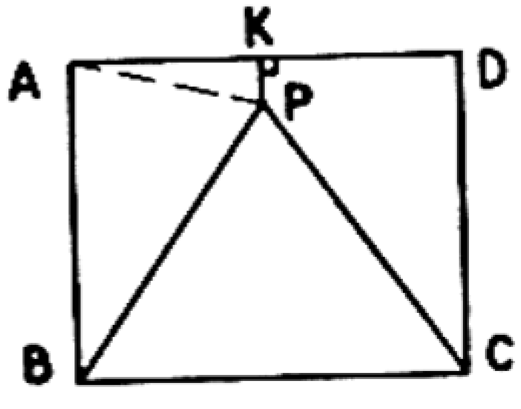

FIG. 27.

4. In Fig. 27, ABCD is a square of side 5 in., and PBC is an equilateral triangle. Find the angles of $\triangle PBA$, the lengths of PA and PK (the perpendicular from P to AD). Check by finding *independently* the perp. from P to BC.

IIB

1. If one side of a triangle is double another, what can you say about the sines of the angles opposite them? And what about the angles themselves?

2. In Fig. 25, sin ABP = 2 sin APB and sin PAN = $\frac{3}{4}$. Show that PN = $1\frac{1}{2}$ AB.

3. From A a perpendicular AN is drawn to a line PQRS and AP, AQ, AR, AS are joined. If NAP, PAQ, QAR, RAS are all 15°, find the ratios AS : AQ and AR : AP.

4. In Fig. 28, AH is perpendicular to BH. Calculate

(i) angle BAC.
(ii) angle B.
(iii) AH.

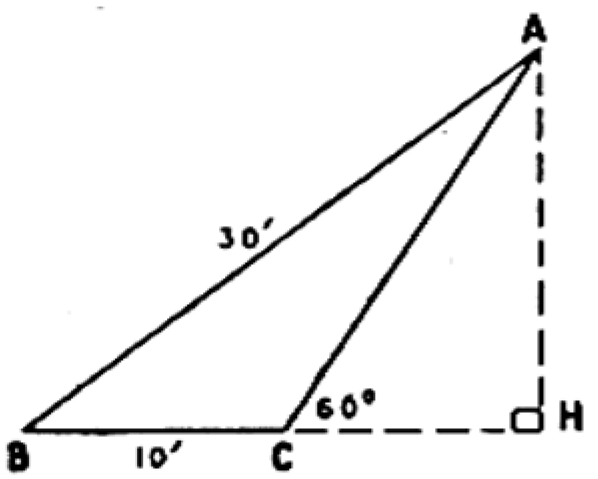

FIG. 28.

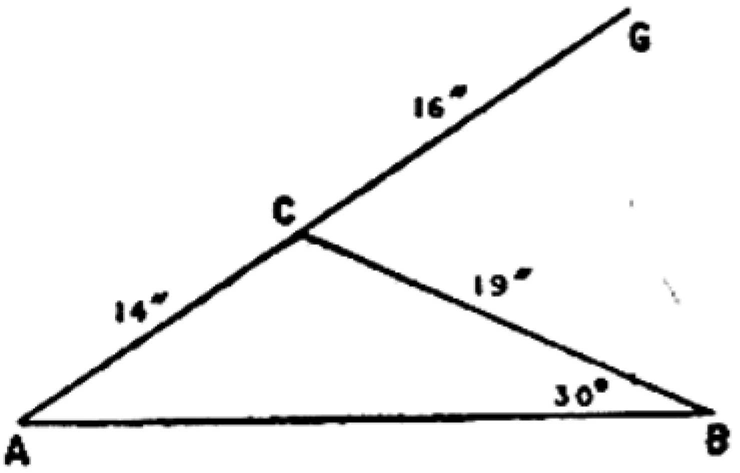

FIG. 29.

IIc

1. In an isosceles triangle PQR, PQ = PR and sin Q = 3 sin P. Find the ratio of the sides.

2. In Fig. 29, find the height of G above AB.

3. The common chord of two circles subtends 110° at the centre of the smaller, whose radius is 9 cm., and 90° at the centre of the larger. Find the radius of the larger circle.

4. A ship steaming due N. at 15 knots observes a lighthouse bearing N.40°E. Two hours later the lighthouse bears N. 74° E. How far is the ship from the lighthouse at the end of the two hours?

(Answer in sea-miles : see IIa, 1 and Fig. 30).

IId

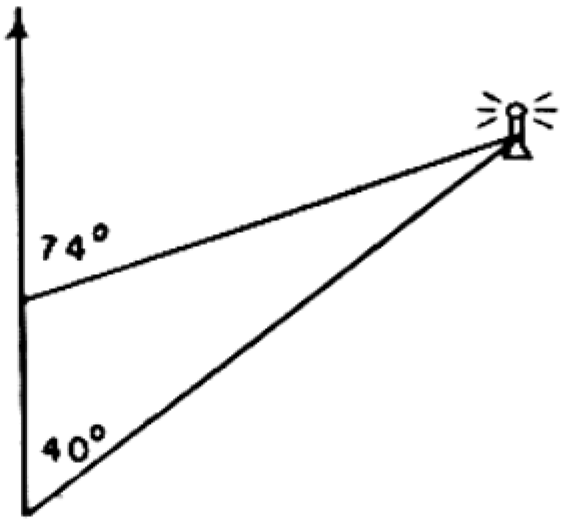

FIG. 30.

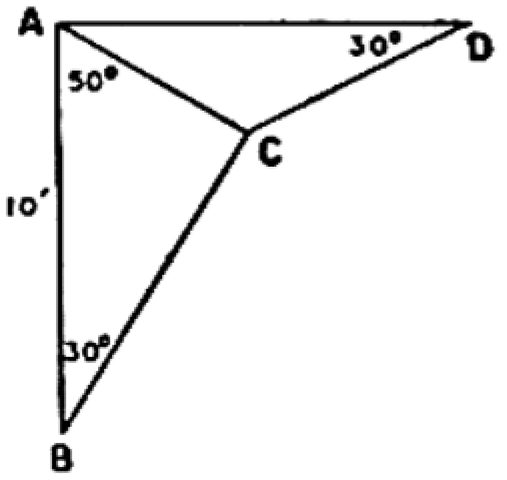

FIG. 31.

1. In Fig. 31, BAD is a right angle. Calculate (i) AC, (ii) AD.

2. A battleship B, steaming due N. at 20 knots, is 15 sea-miles N.W. of a cruiser C. In what direction must the cruiser steam at 30 knots in order to intercept the battleship?

[If they meet at A, then AB/AC = 20/30 and AB/AC = sin ACB/sin ABC. $\angle ABC = 135^\circ$, hence C.]

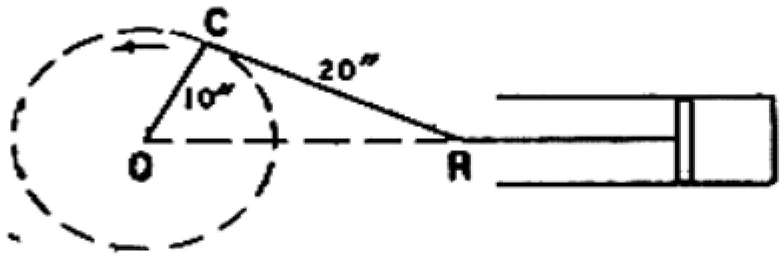

FIG. 32.

3. The crank OC of a steam engine (Fig. 32) is 10 in. long and the connecting rod CR is 20 in. long. Calculate angles CRO and OCR and the distance OR when angle COR has the values 0°, 30°, 60°, 90°, 120°, 150°, 180°. Make a graph showing how OR varies during a complete revolution of the crank.

[Joint work suggested for this exercise.]

CHAPTER III

THE COSINE AND THE COSINE FORMULA

Cosine of Acute Angle

IN Fig. 33, AB is a ladder 15 ft. long resting against a wall and it makes $58^\circ 12'$ with the ground; find AN, the distance of its foot from the wall.

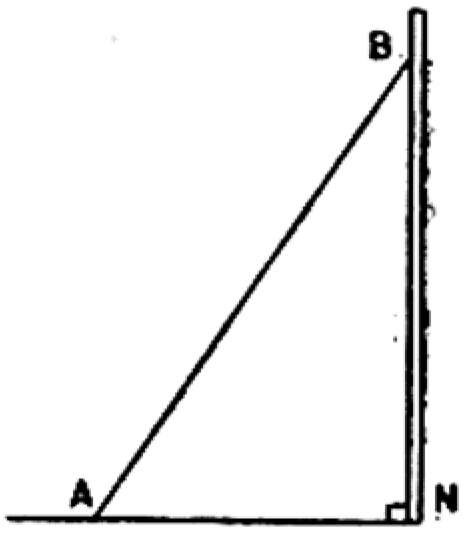

FIG. 33.

If we use the sine table, the first step is to find

$$\angle ABN = 90^\circ - 58^\circ 12' = 31^\circ 48',$$

which is the *complementary* angle to $58^\circ 12'$.

Then

$$AN = AB \sin ABN = 15 \sin 31^\circ 48' = 15 \times \cdot 5270$$

$$\therefore\ AN = 7\cdot 905 \text{ ft.}$$

$$\begin{array}{r} \cdot 5270 \\ 15 \\ \hline 5\cdot 270 \\ 2\cdot 6350 \\ \hline 7\cdot 905 \end{array}$$

Here the sine used is the *sine of the complement of $\angle A$.*

This is shortened into complement-sine of A
cosine of A
cos A.

[This is pronounced koss A.]

To save time a separate table of cosines is made, so that what is wanted is found as $\cos 58^\circ 12'$. [See the table of natural cosines.]

Thus

$$AN = AB \cos A = 15 \cos 58^\circ 12' = 15 \times \cdot 5270 = 7\cdot 905 \text{ ft.}$$

When the angle is that made with the horizontal :—
the sine gives the height, the cosine gives the horizontal distance.

Examples—12

Using Cosines

1. A ladder 20 ft. long makes an angle of 72° with the ground and rests against a wall; how far is its foot from the wall?

2. A balloon is held by a steel cable 3,000 ft. long, approxi-

mately straight and making 57° with the ground. How far horizontally is the balloon from the foot of the cable ?

3. A road slopes upwards at 11° to the horizontal. If I walk 250 yds. along it, how far have I gone horizontally ?

4. A wire stay 50 ft. long is attached to a vertical mast and reaches the ground at P at an angle of 38°18′. How far is P from the foot of the mast ?

5. An aircraft flies 70 miles on a track of 050° in a direction 50° East of North. How far north of its starting point has it flown ?

The Right-angled Triangle Definition of the Cosine

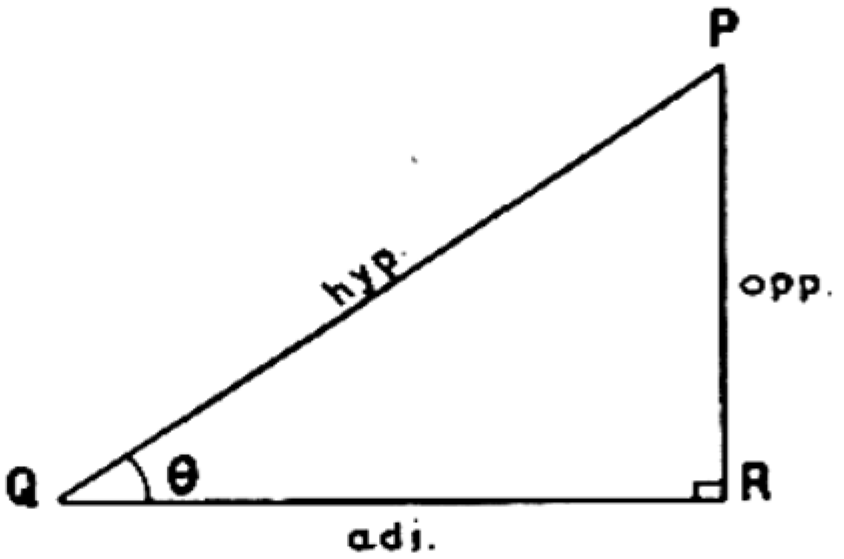

FIG. 34.

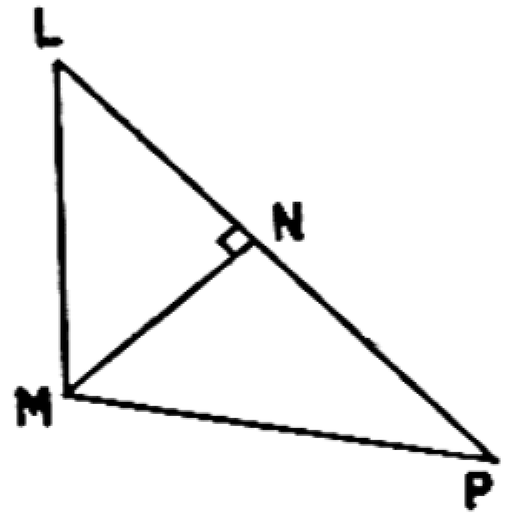

FIG. 35.

In Fig. 34 the side QR is the side *adjacent* to the angle θ,* or for short "adj.," just as PR is "opp." and QP is "hyp."

Thus $\cos\theta = \dfrac{\text{adj.}}{\text{hyp.}}$

As with the sine, this definition works only for acute angles.

In Fig. 35 LN is the side adj. for $\angle$L, so cos L = LN/LM.

Examples—13

Finding Cosines

1. In Fig. 35 write down cos LMN, cos NMP and cos MPN.

2. In the triangle whose sides are 3 in., 4 in., 5 in., what is the cosine of the smallest angle ?

3. An isosceles triangle has sides 7 in., 7 in., 6 in. What is the cosine of a base angle ? [Cut triangle in two, as usual.]

---

* The hypotenuse is also adjacent to the angle, but it has its special name already.

4. By using an equilateral triangle prove that $\cos 60^\circ = \frac{1}{2}$.

5. The sine of an angle is 0·7. Find its cosine by accurate drawing and measurement.

6. In $\triangle ABC$, $a = 9$ cm., $b = 8$ cm., $c = 5$ cm., AD is perp. to BC. Draw accurately; measure BD and CD and so find cos B and cos C.

[	0′	.	42′	48′	54′	1′	2′	3′	4′	5′
66°	·4067	.	3955	3939	3923	3	5	8	11	13]

7. Use the above excerpt from the table of natural cosines to find cos 66°50′ and cos 66°58′.

[Sol'n. For cos 66°50′ take 66°48′ and *subtract* what is below 2′, result ·3939 − ·0005 = ·3934.]

8. Use the table of natural cosines to find cos 19°33, cos 78°10′ and cos 2°43′.

Finding Angles from Cosines

9. What are the angles whose cosines are ·9304, ·4274, $\frac{1}{2}$ cos 6°18′ ?

10. What are the angles whose cosines are ·9581, ·8315, ·1775, $\frac{1}{2}$ cos 30° ?

[Sol'n. ·9581 lies between the cosines of 16°36′ and 16°42′ and is less by ·0002 than the former. 2 is under both 2′ and 3′ in the difference columns; so the answer is 16°38′ or 16°39′.]

11. With the data of Ex. 3 find the base angles.

12. OA is the radius of a circle, centre O, and is trisected at M, N. Lines MP, NQ perp. to OA meet the circle at P and Q. Find the angles AOP, AOQ.

13. OA, OP are two radii of a circle such that ∠AOP = 80°. Find a point Q on the arc AP such that cos AOQ = 2 cos AOP. Measure ∠AOQ. Also calculate it, using tables.

14. A lean-to shed has a floor 8 ft. by 4 ft. and a roof 8 ft. by 6 ft. (inside measurements). What angle does the roof make with the horizontal ?

15. Each wing of an aircraft is 50 ft. long and when the tip is 18 ft. above the ground the root is 15 ft. 6 in. above the ground. What is the angle between the wings ? (Fig. 36.)

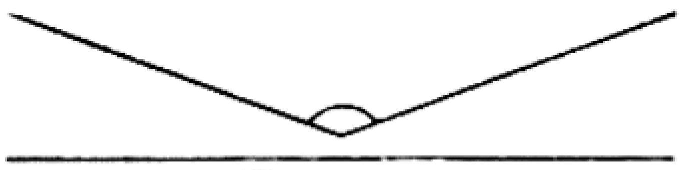

FIG. 36.

Greater Angle, Smaller Cosine

16. The sides of a triangle are $a = 5$ in., $b = 8$ in., $c = 6$ in. Arrange cos A, cos B, cos C in order of ascending magnitude.

17. Divide the acute angles into those whose cosines are greater or less than $\frac{1}{2}$.

18. Two isosceles △s. have the equal sides 10 in. long. In one the base is 18 in., in the other 6 in. Find the cosines of the base angles and the angles themselves.

19. A circle has a diameter AB 10 ft. long. AC is a chord making an angle θ with AB. What is the length of AC? What happens to AC as θ grows from 0° to 90°?

20. Explain why the numbers in the difference columns of a cosine table must be subtracted. Is this true in a table of logarithmic cosines?

Using Cosines

21. The equal sides of an isosceles triangle are each 25 in. and a base angle is 47°10′. Find the length of the base.

22. A regular pentagon is inscribed in a circle of radius 5 cm. Find the perpendicular distance from the centre to one of the sides.

23. In walking 725 ft. at a uniform slope I rise 50 ft. What is the angle of slope and what is the horizontal distance?

24. The pendulum of "Big Ben" is 13 ft. long. How high does its tip rise if it swings 10° each side of the vertical?

25. A cubical box, side 3 ft., is tilted through 30°. Show that its highest edge is $3(\cos 30° + \sin 30°)$ above the ground, and work this out.

Cosines and Projections

The cosine is especially important in giving the *projection* of one line on another line. Thus in Fig. 37, if PM, QN are perp. to AB, then M is the projection of P, N the projection of Q, and MN the projection of PQ.

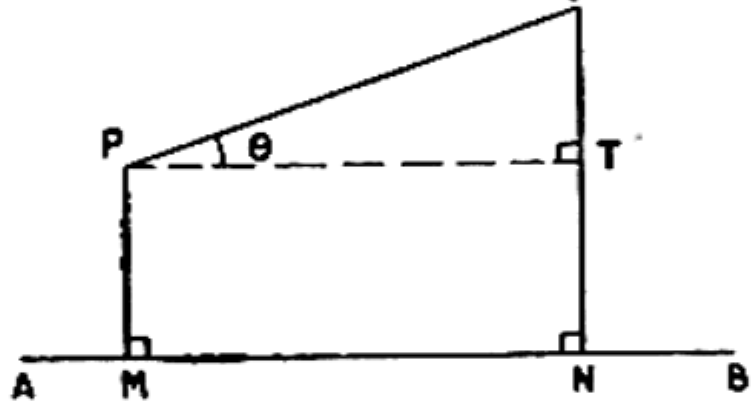

FIG. 37.

Here MN = PT = PQ cos θ ;
thus the *projection of a length r on a line making an angle θ with it is*

$$r \cos \theta.$$

If the diagram (Fig. 37) is a map in the normal position and PQ represents a distance travelled in a direction $\theta°$ north of east, then MN or PQ cos θ represents the distance gone towards the east.

Examples—14

1. PQ is 4 in. long and makes 25° with AB. What is its projection on AB ? Does it make any difference (i) if **P** lies on AB, (ii) if P,Q are on opposite sides of AB ?

2. Jack walks 200 yds. at 33° N. of E., then 150 yds. due E., then 250 yds. at 20° S. of E. How far east is he of his starting point ?

3. Bill rides 9 miles N.E. and then 4 miles N.70°E. How far north has he got ? How far east ?

4. In $\triangle ABC$, $AB = 4{\cdot}2$ in., $\angle B = 43°36'$, $BC = 6{\cdot}3$ in. AN is perp. to BC. Find BN and CN.

5. A circle of radius 10 cm. whose centre is at the origin O cuts the axes at A and B. The arc AB is divided into five equal parts by radii OP_1, OP_2, OP_3, OP_4. Find the projections of these radii on OA.

Cosines and Forces

Fig. 38 represents a truck being pulled along rails by an oblique force of magnitude P. It is found by experiment that the effect of the force P along AB is P cos θ. This is an important property of the cosine.

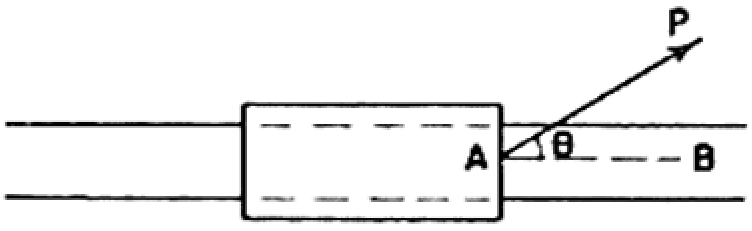

FIG. 38.

The effect of a force P at an angle θ with its direction is P cos θ.

Examples—15

1. Two men pull a cart with ropes. One rope makes 11° with the direction of motion, the other 14°. If each pull is 50 lb., what is the combined effect in the direction of motion?

2. A truck rests on rails. Robert tries to pull it forward by a tug of 80 lb. on a rope making 20° with the rails; Richard to pull it backward by a tug of 160 lb. on a rope making 40° with the rails. Which way does the truck move? If the 40° had been 65°, which way?

3. A weight is supported on two ropes which make 27° and 39° with the vertical. If the tensions of the ropes are 39 lb. and 27 lb. respectively, what is the weight? If free to move sideways would it do so?

4. What percentage of a force is wasted if it has to be applied at 30° to the desired direction?

5. A horse walking on the towpath pulls a barge along a canal. If the rope makes 15° with the bank by what percentage must the horse's effort exceed the direct pull needed to keep the barge moving.

Bearings

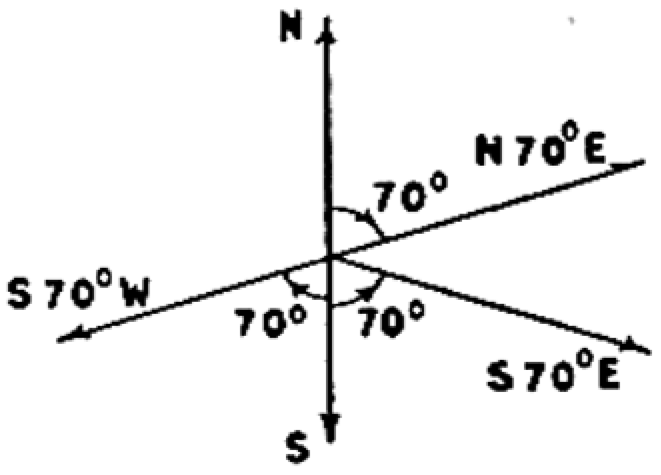

FIG. 39.

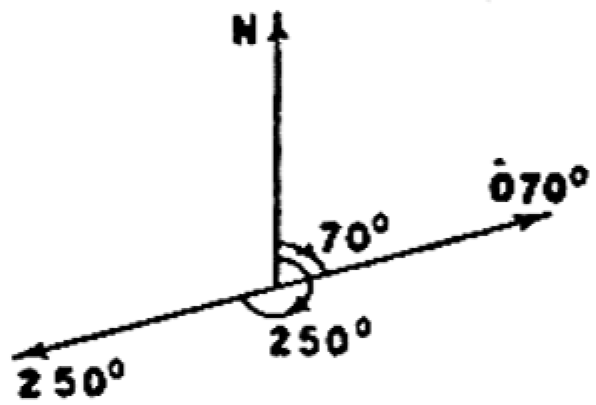

FIG. 40.

The figures show two ways in which *bearings*—giving the direction of a line—are measured.

In Fig. 39 reference is made to North or South, whichever is nearer. This is the civil system.

In Fig. 40 a three-figure angle is named, measured clockwise from North. This is the military system.

Examples—16

1. I walk a mile N. 70° E. How far am I north of my starting point and how far east of it.

2. After bicycling 12 miles S.50°W. Harry guesses that he is 8 miles south and 9 miles west of where he started. By how much is he wrong ?

3. P is 6000 yds. from A on a bearing 057. How far is P north of A ?

4. If the bearing of Q from B is 162°, how far is Q south of B if the distance from B to Q is 2 miles ?

5. An aircraft flies for 30 minutes on a course 123°, and then for 20 minutes on a course 231°, keeping a steady speed of 240 m.p.h. How far is it south of its starting point ?

Cosine of Obtuse Angle

To go a distance r from O in a direction making $\theta°$ with a line OA to the east is to go a distance $r \cos \theta$ to the east of O, if θ is acute. If θ is obtuse you do not go to the east at all but to the west, travelling a distance $r \cos \text{POB} = r \cos (180° - \theta°)$.

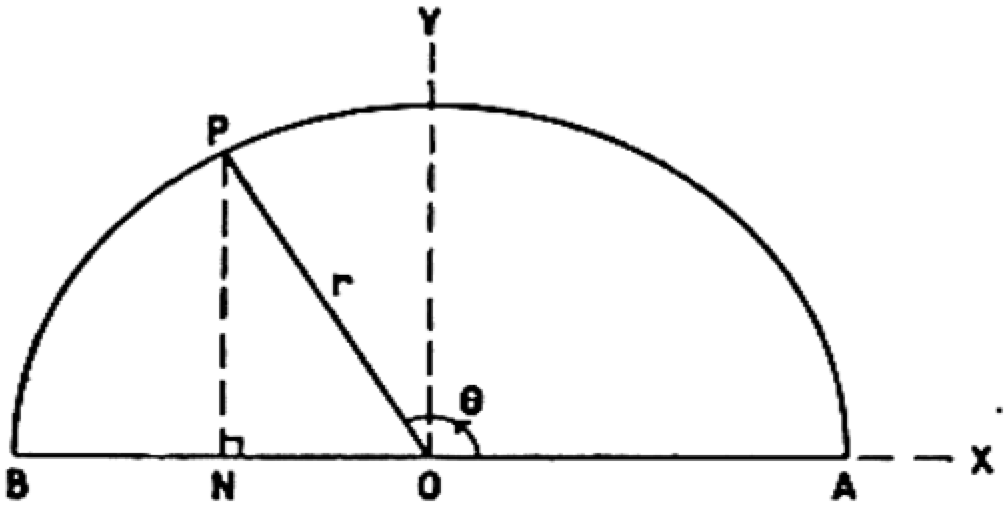

FIG. 41.

A force R which pulls along a line making an acute angle θ with OA has an effect $R \cos \theta$ along OA ; but if θ is obtuse there is no effect along OA but an effect in the *opposite* direction, $R \cos \text{POB} = R \cos (180° - \theta)$.

For these reasons, among others :—

If θ is obtuse, cos θ is defined to be $-\cos (180 - \theta)$.

Thus $\cos 140° = -\cos 40°$

$\cos 165° = -\cos 15°$, and so on.

Contrast $\cos \theta = -\cos (180° - \theta)$

with $\sin \theta = \sin (180 - \theta)$.

Using Coordinates

If $\angle XOP = \theta$, $OP = r$, and the coordinates of P are x, y,

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

Note that if θ is obtuse, as in Fig. 41 above. the "x" of P is negative, while its "y" is positive, so that $\cos\theta$ is negative, while $\sin\theta$ is positive. These definitions can be used for angles greater than 180°; see pp. 113, 114.

Examples—17

1. Use the tables to find cos 135° 18′ and cos 170° 12′.

2. If $\cos 35° + \cos\theta = 0$, what angle is θ?

3. What kind of triangle is $\triangle ABC$ if cos C is negative?

4. Find θ by drawing and measurement (i) if $\cos\theta = -\cdot 4$, (ii) if $\cos\theta = -\cdot 8$.

5. Show that the equation $\cos\theta + \cos(180° - \theta) = 0$ is satisfied both if $\theta = 90°$ and if $\theta = 0°$.

6. Show that in Fig. 41 $NA = r - r\cos\theta$ whether θ is acute, right or obtuse.

Pythagoras in Trigonometry

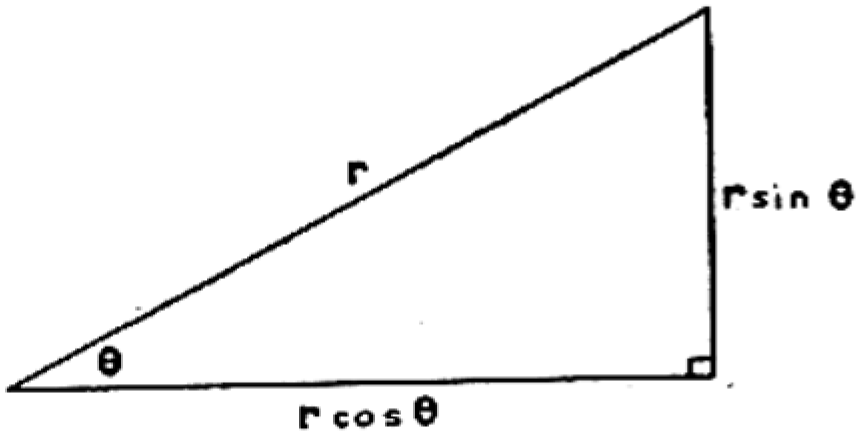

FIG. 42.

Fig. 42 shows how the sine and cosine can be used as multipliers to find lengths. The height is $r\sin\theta$ and the base $r\cos\theta$.

Pythagoras' theorem gives $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$.

It is usual to write $\cos^2\theta$ for $(\cos\theta)^2$ and $\sin^2\theta$ for $(\sin\theta)^2$, so the equation is written $r^2\cos^2\theta + r^2\sin^2\theta = r^2$.

Dividing both sides by r^2, the trigonometrical version of Pythagoras' theorem is obtained, $\cos^2\theta + \sin^2\theta = 1$.

This must be learnt by heart.

Examples—18

1. Test the accuracy of working with four-figure tables by finding what $\cos^2\theta + \sin^2\theta$ comes to in decimals (i) if $\theta = 34°17'$, (ii) if $\theta = 80°54'$.

2. Use $\cos^2\theta + \sin^2\theta = 1$ to work out $\sin\theta$ (i) if $\cos\theta = \frac{3}{5}$, (ii) if $\cos\theta = \frac{12}{13}$.

3. If for an acute angle $\sin\theta = 0{\cdot}3$, find $\cos\theta$, (i) by tables, first finding θ; (ii) by direct calculation.

4. Prove that $(x + y\cos\theta)^2 + (y\sin\theta)^2$

$$= x^2 + y^2 + 2xy\cos\theta.$$

5. Prove that $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2$.

6. In Fig. 43, AD is perp. to BC,

prove that $\mathrm{CD} = a - c\cos\mathrm{B}$,

$\mathrm{BD} = a - b\cos\mathrm{C}$,

$a = b\cos\mathrm{C} + c\cos\mathrm{B}$.

7. Using Fig. 43, prove that $c^2 - b^2 = c^2\cos^2\mathrm{B} - b^2\cos^2\mathrm{C}$.

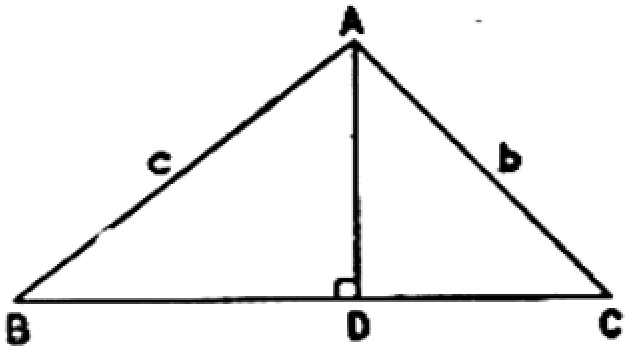

FIG. 43.

8. In $\triangle$ABC, BE,CF are perpendiculars from B, C to the opposite sides. By finding two expressions for cos A prove that BA.FA = CA.EA.

9. Prove that $\cos^2\mathrm{B} - \cos^2\mathrm{A} = \sin^2\mathrm{A} - \sin^2\mathrm{B}$.

10. What is the complement of $(45° + \theta)$?

Prove that $\sin^2(45 + \theta) + \sin^2(45° - \theta) = 1$.

The Concealed Minus

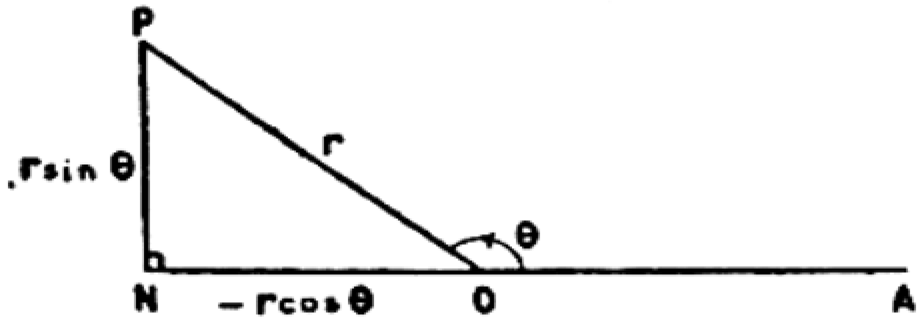

FIG. 44.

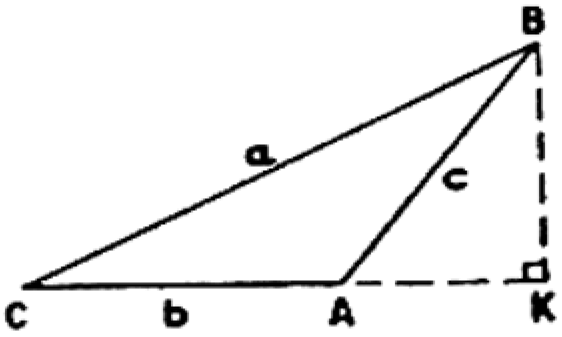

FIG. 45.

It seems curious to mark a length $-r\cos\theta$ as in Fig. 44.* but if θ is obtuse, $\cos\theta$ contains a concealed minus and $-r\cos\theta$ is $+$.

---

* In the words of the poet :

> "Remember the sine gives the height
> And the cosine the step to the right."

Examples—19

1. In Fig. 45 prove that CK $= b - c \cos A$.

2. Is the result positive or negative in
(i) $\cos 40° + \cos 150°$; (ii) $\cos 50° + \cos 140°$;
(iii) $\cos 30° + \cos 130°$; (iv) $\cos 40° + \cos 140°$?

3. In Fig. 45 a perpendicular CN is drawn to BA produced; what is the length of BN? Prove that BN—CK is positive if $c > b$.

4. **Use Fig. 44 to prove that $\cos^2\theta + \sin^2\theta = 1$, if θ is obtuse.**

5. In △ABC, AD is the perp. to BC from A. Prove that BD $= a - b \cos C$ whether ∠C is acute or obtuse.

Miscellaneous

6. Given the cosine of an angle, show how to find the angle by drawing and measurement without using tables. Find thus the angles whose cosines are (i) ·7, (ii) — ·45.

7. An isosceles triangle has sides 8 in., 8 in., 5 in. Find the cosine of one of the equal angles, and the angle itself.

8. If $\theta = 70°$ and $\cos \phi = 2 \cos \theta$, find ϕ.

9. *Explain how to find cosines from a table of sines. [Often this must be done because no tables of cosines are provided.]* Find in this way cos 14°36′ and cos 112°6′.

10. Given that $\sin 30° = \frac{1}{2}$ and $\sin 60° = \frac{1}{2}\sqrt{3}$, find the numerical value of $2(\sin 120° + \cos 120°)$.

11. Find two values of $\cos \theta$ if $\sin \theta = ·3$.

12. The cosine of the base angle of an isosceles triangle is ·48; find the cosine of the vertical angle. [First find the angle itself.]

13. A pendulum 39 in. long swings through 14° on each side of the vertical. How high above its lowest position does its end rise?

14. ABCD is a rhombus of four bars, each 10 in. long, with pivots at A, B, C, D. What change in angle ABC is caused when BD is changed from 18 in. to 10 in.? What is the change in AC?

15. Find the lengths of the altitudes of an isosceles triangle in which the equal sides are 5 in. long and include an angle of 36°.

The Cosine Formula

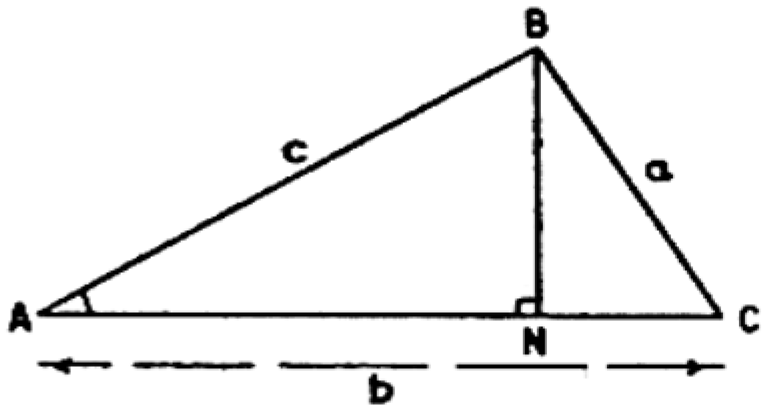

FIG. 46.

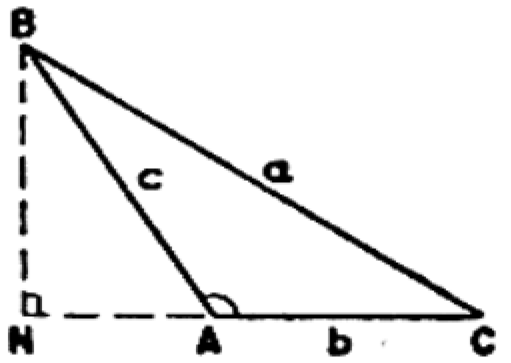

FIG. 47.

In Fig. 46 AN = c cos A. In Fig. 47 AN = — c cos A for it = c cos (180° — A).

$\therefore$ In both figures CN $= b - c \cos A$, and also in both figures BN $= c \sin A$.

Hence by Pythag. for $\triangle$BCN

$$\begin{aligned} a^2 &= (b - c\cos A)^2 + (c\sin A)^2 \\ &= b^2 - 2bc\cos A + c^2\cos^2 A + c^2\sin^2 A, \end{aligned}$$

but $c^2\cos^2 A + c^2\sin^2 A = c^2$, since $\cos^2 A + \sin^2 A = 1$.

$$\therefore \mathbf{a^2 = b^2 + c^2 - 2bc \cos A.}$$

This is the Cosine Formula. It is often called the Cosine Rule.

The cosine formula is second in importance only to the sine formula. It must be known thoroughly.

Examples—20

1. There are formulæ for b^2 and c^2 similar to that for a^2. Write them down from symmetry.

2. Show that the cosine formula may be written

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

and write down from symmetry the corresponding formulæ for cos B and cos C.

3. From the formula of No. 2 show that A is obtuse if $b^2 + c^2 < a^2$.

4. Show that in Fig. 46 the result may be written

$$BC^2 = BA^2 + AC^2 - 2AC\,.\,AN$$

and that in Fig. 47 it may be written

$$BC^2 = BA^2 + AC^2 + 2AC\,.\,NA.$$

[When written in these ways the results are called *the extensions of Pythagoras Theorem*. The cosine formula is easier to remember, but the "extensions" show more clearly that if A is obtuse you must add, not subtract.]

Cosine Formula

5. In $\triangle ABC$, $\cos A = 0.7$, $b = 7$ in., $c = 6$ in.; find a.
[Sol'n. $a^2 = b^2 + c^2 - 2bc \cos A = 49 + 36 - 84 \times 0{\cdot}7$
$= 85 - 58{\cdot}8 = 26{\cdot}2$.
$\therefore\ a = \sqrt{26{\cdot}2} = 5{\cdot}119 \simeq 5{\cdot}12$ in.]

6. In $\triangle ABC$, $\cos A = 0{\cdot}6$, $b = 5$ cm., $c = 7$ cm.; find a.

7. In $\triangle ABC$, $\cos B = -0{\cdot}4$, $a = 6$ in., $c = 4$ in.; find b.

8. The sides of a triangle are 6 in., 7 in., 10 in. Find the cosines of the smallest and largest angles and hence find these angles.

[Sol'n. First cosine $= \dfrac{7^2 + 10^2 - 6^2}{2 \times 7 \times 10} = \dfrac{113}{140} = 0{\cdot}8071$
$\therefore$ angle $= 36°11'$.

Second cosine $= \dfrac{6^2 + 7^2 - 10^2}{2 \times 7 \times 7} = -\dfrac{15}{98} = -0{\cdot}1786$
$\therefore$ angle $= 180° - 79°\,43' = 100°\,17'$.]

9. Without using the results of Ex. 8, find the cosine of the third angle of the same triangle and hence find the angle. Check, using angle-sum.

10. Find the largest angle of the triangle whose sides are 6 in., 7 in., 8 in.

11. The sides of a triangle are 3 cm., 4 cm., 6 cm. Find the cosines of the angles and the angles themselves.

12. Find cos 60° by applying the cosine formula to the triangle whose sides are each x in.

13. In a triangle sides 4 in. and 7 in. contain an angle of 60°. Find the third side.

14. An isosceles triangle has an angle 120°. Show that the square of the largest side is three times the square of one of the other sides.

15. Given $\cos 30° = \frac{1}{2}\sqrt{3}$, find the base of an isosceles triangle whose equal sides are each 10 cm. and contain an angle of 30°. By bisecting the base, prove that
$$\sin 15° = 0{\cdot}2588.$$

16. If in $\triangle ABC$, B and C are acute and AD is the perpendicular from A to BC, write down the results given by the extensions of Pythagoras' Theorem for (i) AB^2, (ii) AC^2.

Prove that $AB^2 - AC^2 = BC(BD - CD)$.

17. If in $\triangle ABC$, B is acute and C obtuse and AD the perpendicular from A to BC, write down the results given by the extensions of Pythagoras' Theorem for (i) AB^2, (ii) AC^2. Prove that $AB^2 - AC^2 = BC(BD + CD.)$

18. Apply the cosine formula to the triangle whose sides are a, b, b, to prove that $a^2 = 2b^2 (1 - \cos A)$. Prove also, by drawing the bisector of $\angle A$ that $a = 2b \sin \frac{1}{2}A$. Deduce a relation between $\cos A$ and $\sin \frac{1}{2}A$. Change this into the form $2 \cos^2 \frac{1}{2}A = 1 + \cos A$.

19. If in a triangle $a = 12{\cdot}10$ in., $b = 14{\cdot}60$ in., $c = 17{\cdot}30$ in., show that $b^2 + c^2 - a^2$ by use of square tables is known to four figures but $a^2 + b^2 - c^2$ to three figures only. Which will be found more accurately by the cosine formula, $\cos A$ or $\cos C$?

20. In the parallelogram ABCD, $AB = x$, $AD = y$; prove that $AC^2 = x^2 + y^2 + 2xy \cos A$.

The Pyramid and Sphere

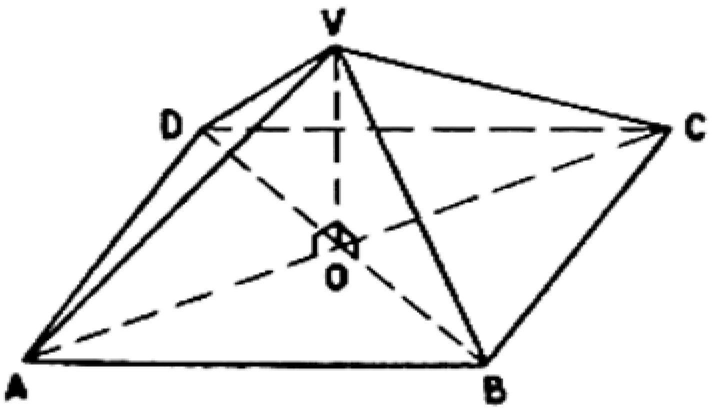

FIG. 48.

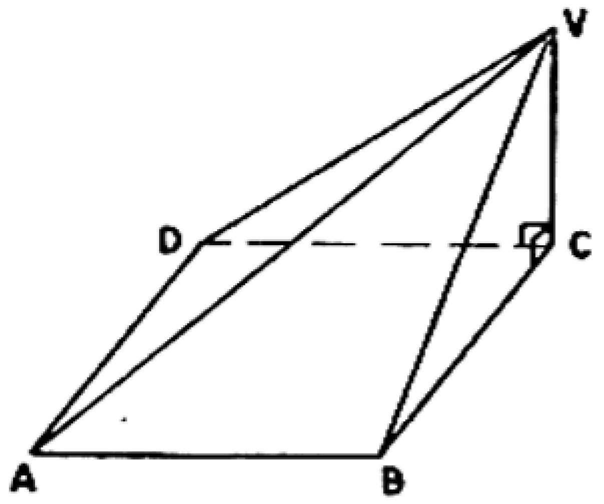

FIG. 49.

Note on drawings

Fig. 48 represents a (symmetrical) pyramid on a square base.

A parallelogram ABCD is drawn first for the base; then the diagonals are put in. Then the height OV, upright from O, is drawn and the figure completed.

Fig. 49 represents a pyramid with the vertex above one corner of the base.

Examples—21

1. In Fig. 48, each side of the base is 3 in., each slant edge 5 in. Prove that $AC \bumpeq 4{\cdot}242$ in. and find the angle which VA makes with AC (use AO). Find also $\angle AVB$ by the cosine formula. Check by bisecting triangle VAB.

2. In Fig. 49, ABCD is horizontal. Which faces are vertical and which angles in the figure are right angles ? If each side of ABCD is 4 in. and VC is 3 in., find the slant edges and $\angle AVB$. (Remember that a vertical line is at right angles to all horizontal lines which meet it.)

3. A rectangular block stands on its smallest face which is 4 in. by 3 in. and the block is 12 in. high. Prove that the diagonal of the block is 13 in. and find the angles which this makes with the diagonals of each of the three faces, *i.e.*, the angles it makes with these faces.

4. ABC is an equilateral triangle of sides 5 in. and VA is at right angles to the plane ABC. If VB = 7 in., find $\angle$VBA. If N is the mid point of BC, find VN and AN and hence find the angle which the plane VBC makes with ABC.

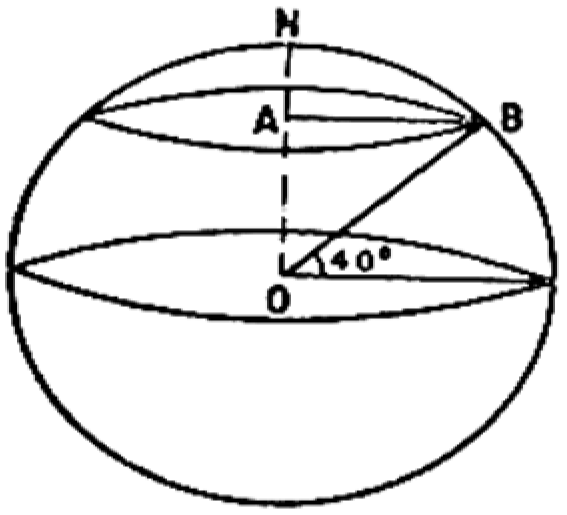

FIG. 50.

5. Taking the earth to be a sphere of radius 4,000 miles, find the radius of the 40th parallel of latitude (AB in Fig. 50).

6. In Fig. 50, find the distance from B to the plane of the equator, *i.e.*, the perpendicular distance from B to OC (OB = 4,000 miles).

7. In a sphere of radius R and if in Fig. 50 $\angle$BOC is called θ, find trigonometrical expressions for AO, AB, AN.

8. If, in Fig. 49, ABCD is a horizontal square whose side is 4 in., and VC = 7 in., calculate VA and the angle AVC (the angle between VA and the vertical).

9. In Fig. 48, ABCD is a square whose side is 12 ft., and VC = 11 ft. Calculate VO and the angle AVO.

10. Make a sketch of the square pyramid shown in Fig. 48, with M, the mid-point of BC, joined to O and V. Since the triangles OBC, VBC are isosceles OM and VM are perpendicular to BC. The angle OMV is then the angle between the *face* VBC and the *base* ABCD.

Calculate this angle (i) when AB = 10 ft., VB = 13 ft.
(ii) when AB = 12 ft., VB = 10 ft.

11. If, in Fig. 48, ABCD is a horizontal rectangle with AB = 18 in., VB = 17 in., and BC = 16 in., calculate (i) the angle between VBC and the base, (ii) the angle between VB and the vertical.

[Calculate (i) VM and OM as in No. 10, (ii) calculate VO.]

12. A pyramid VABC has a triangular base ABC. If VA = 8 in., VB = VC = 5 in., and angles AVB, BVC, CVA are all 60°, calculate (i) the sides, (ii) the angles of $\triangle$ABC.

[Use the cosine formula on $\triangle$VAB.]

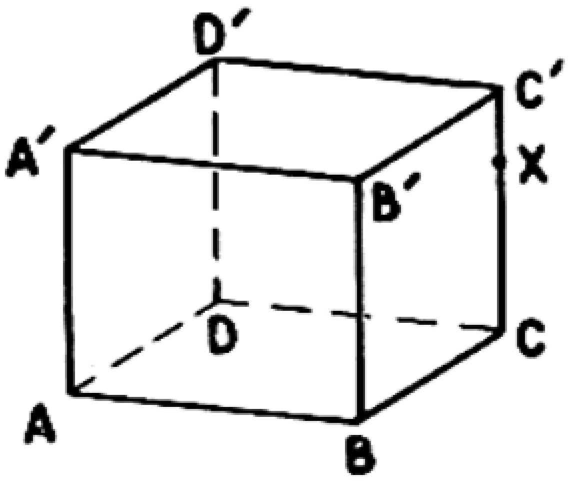

FIG. 51.

13. Fig. 51 represents a rectangular box (a brick, a cuboid, and a rectangular parallelepiped are also of this shape *).

If AB = 6 ft., BC = 8 ft. = BB′, and CX = 6 ft., calculate the lengths of A′X, A′B and BX and hence find the angle A′BX. Find also the angle between A′X and the back of the box (CDD′C′).

14. A pyramid stands on a regular hexagonal base, the sides of the hexagon being 10 in. long. If the slant edges of the pyramid are 15 in. long, find the angle between a slant edge and the horizontal hexagonal base.

15. A pyramid VABC has a triangular base ABC with BC = 12 in., AB = AC = 10 in. = VB = VC, and VA = 6 in. Find the angle between the face VBC and the base. [This will be ∠VMA, where M is the mid-point of BC.]

16. A pyramid VANB has its base ANB right angled at N, and VN is perpendicular to the base. If VA = 20 ft. = VB, and ∠VAN = 45° = ∠VBN, calculate AB and the angle between the face VAB and the base. [Show that △VAN ≡ △BAN; find MN where M is mid-point of AB.]

Advantages and Disadvantages of the Cosine Formula

Solution of triangles—determining the other parts, if three parts are given—is completed in theory when the use of the Cosine Formula is added to that of the Sine Formula.

The S.S.S. case is soluble because the cosine of any angle can be found and the S.A.S. case is soluble because the remaining side can be found.

In practice, however, there are objections to the use of the cosine formula; (i) though it is easy to use if the sides are small whole numbers, it is not easy to use if they are given to three or four figures, since there are then three terms to be found separately (b^2 and c^2 by square tables and $2bc \cos A$ by logs) and then to be combined.

(ii) Though in $b^2 + c^2 - 2bc \cos A$ the separate terms may be accurate to four figures, the subtraction will often reduce this to three figures. The same is true of $b^2 + c^2 - a^2$. The result is that formulæ more difficult to prove but easier and more accurate in use are generally employed. These are discussed in Chapter VI.

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* *Cf.* Lewis Carroll: "Farewell, old brick or rather let me say, old parallelepiped."

If however these other formulæ are not known or are forgotten, the cosine formula can be used as in the following examples, which need the use of logarithms and tables of squares and square roots.

Examples—22

Solution of Triangles by Cosine Formula

Case S.A.S. 1. The sides 27·3 yds. and 42·6 yds. contain an angle 47°9′. Find the remaining sides and angles.

[Sol'n. Call the given parts b, c, A ; a is found by the cosine formula and the solution completed by the sine formula.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$= 27{\cdot}3^2 + 42{\cdot}6^2 - 2 \times 27{\cdot}3 \times 42{\cdot}6 \cos 47°9'.$$

Use logs for the last term

Logs		Squares
0·3010	$a^2 = 27{\cdot}3^2 + 42{\cdot}6^2 - 1582$	745·3
1·4362	$= 978$	1815
1·6294	$a = 31{\cdot}27$	2560
$\bar{1}{\cdot}8326$		1582
3·1992	$\simeq 31{\cdot}3$ yds.	978

$\frac{\sin B}{27{\cdot}3} = \frac{\sin 47°9'}{31{\cdot}27}$	$\bar{1}{\cdot}8651$	$\frac{\sin C}{42{\cdot}6} = \frac{\sin 47°9'}{31{\cdot}27}$	Logs
	1·4952		$\bar{2}{\cdot}3699$
B = 39°47′	2·3699	C = 86°42′ to 48′	1·6294
A = 47°9′	1·4362		$\bar{1}{\cdot}9993$
C = 93°4′	$\bar{1}{\cdot}8061$	or its supplement.	

]

2. Solve the △ given

(i) $a = 9{\cdot}62$ in., $b = 7{\cdot}41$ in., C = 47°42′.
(ii) $c = 3{\cdot}25$ in., $a = 5{\cdot}17$ in., B = 104°16′.

Case S.S.S. 3. Find the angles if the sides of a triangle are 12·1 in., 14·6 in., 17·3 in.

[Sol'n. Call the sides a, b, c, in the order named ;

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

	Squares	Logs	
$\therefore \cos A = \frac{14{\cdot}6^2 + 17{\cdot}3^2 - 12{\cdot}1^2}{2 \times 14{\cdot}6 \times 17{\cdot}3}$	213·2	0·3010	2·5636
	299·3	1·1644	2·7034
A = 43°33′	512·5	1·2380	$\bar{1}{\cdot}8602$
	146·4	2·7034	
	366·1		

]

Though the cosine formula can of course be used to find cos B, it is found more quickly by the sine formula,

$$\frac{\sin B}{b} = \frac{\sin A}{a} \qquad \sin B = \frac{14{\cdot}6 \sin 43°33'}{12{\cdot}1}$$

$$B = 56°14'$$

$$\text{whence } C = 80°13'$$

Logs
$1{\cdot}1644$
$1{\cdot}0828$
$0{\cdot}0816$
$\bar{1}{\cdot}8382$
$\bar{1}{\cdot}9198$

To check, either work sin C from $\frac{\sin C}{c} = \frac{\sin A}{a}$ or cos C from the sides.]

4. Prove that the checks suggested above give

$$\log \sin C = \bar{1}.9934 \quad C = 80°0'$$
$$\log \cos C = \bar{1}{\cdot}2321 \quad C = 80°11'.$$

[This example gives a good instance of the inaccuracy to which four-figure tables may lead. The difference between log sin 80°0′ and log sin 80°13′ is only 0·0002. Solution by the formulæ of Chapter VI gives the angles 43°34′, 56°16′, 80°10′.]

5. Find all the angles if the sides are (i) 13·4 in., 15·0 in., 18·9 in. ; (ii) 54 yds., 47 yds., 32 yds. ; (iii) 8·7 in., 9·6 in., 14·4 in.

6. Find the diagonals of a parallelogram whose sides are 16·3 in. and 24·7 in. and of which one angle is 67°38′.

7. AM is median of ⊿ABC. If BC is 42 cm., AN is 20 cm. and the acute angle between them is 39°30′, find AB, AC and the angles of the figure.

The Double Cosine Formula $a = b \cos C + c \cos B$

8. In Fig. 52, find the length of BD from c and B in △ABD and of CD from b and c in △ACD.

Hence prove $a = c \cos B + b \cos C$.

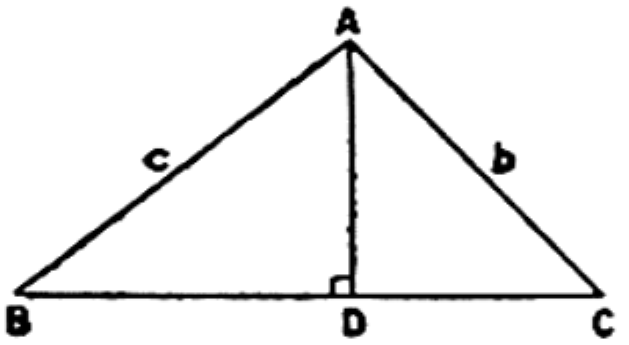

FIG. 52.

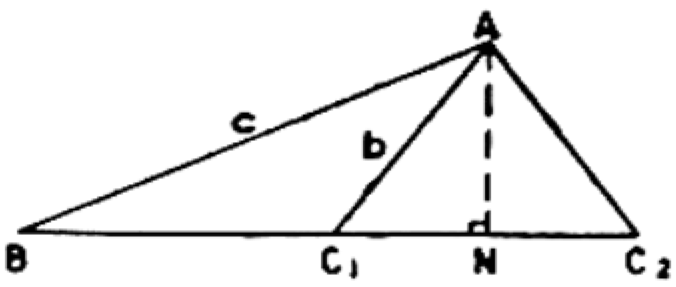

FIG. 53.

9. Prove that the formula of Ex. 8 is still true if C is obtuse.

10. In Fig. 53, which shows the two triangles for the A.S.S. case in which b, c, B are given, note that the two values of a are BN $\pm$ CN and show that the double cosine formula gives both these values by using the two values of C.

[The double cosine formula has five quantities in it (not four as the sine and cosine formulæ have), so it is no use for *starting* the solution of a triangle with three things given.]

REVISION EXERCISES

IIIA

1. A triangle ABC has AB $= 10$ in., BC $= 6$ in., angle ABC$=30°$ and $\angle$BCA is acute; BD is the perp. from B to AC. Calculate AD, BD, $\angle$ACB and find the area of the triangle.

2. London is in latitude $51\frac{1}{2}°$ N. approximately. If the earth's radius is taken as 4,000 miles, how far is London from (i) the earth's axis, (ii) the plane of the equator?

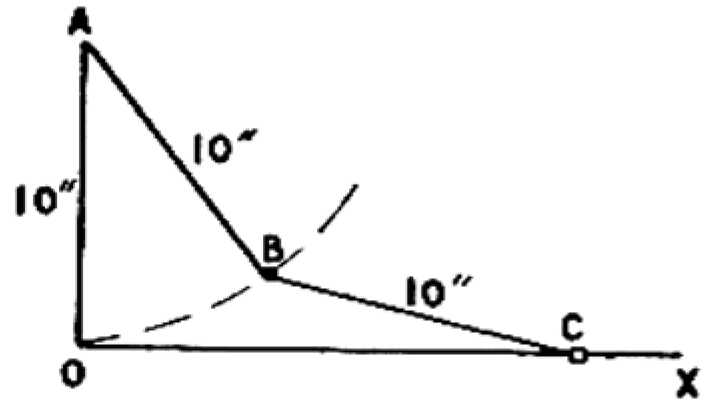

FIG. 54.

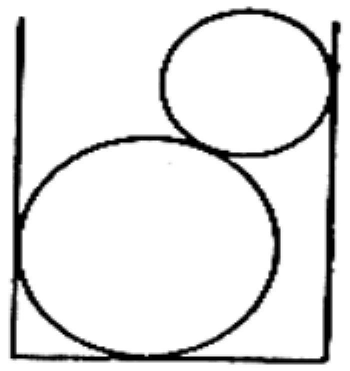

FIG. 55.

3. The rods AB, BC in Fig. 54 are pivoted at A and B, while C moves along OX. Angle AOX is a right angle. What alteration in angle BCO is caused when angle OAB changes from 20° to 40°, and what is the alteration in OC?

4. Two wheels, diameters 5 ft. and 3 ft., are packed in the end of a truck 6 ft. wide. What is the inclination, to the horizontal, of the line joining the centres? (Fig. 55.)

IIIB

1. ABC is an equilateral triangle inscribed in a circle of radius 10 cm. Calculate BC and the altitude from A.

2. ABCD is a trapezium, with AB parallel to DC. The lengths of AB, BC, CD, DA are 28, 13, 14, 15 in. respectively. Calculate the angles of the trapezium. [Draw a parallel to BC through D.]

3. A triangle ABC is obtuse-angled at C. The perpendicular from A to BC meets BC, produced, at D. CA, CB, CD are 10, 20, 3 in. respectively. Calculate the angle ACB and the side AB.

4. A parallelogram has sides 5 in. and 10 in. and acute angles 60°. Calculate the lengths of the diagonals.

IIIc

1. A regular pentagon has sides 7 cm. long. Find the length of a diagonal.

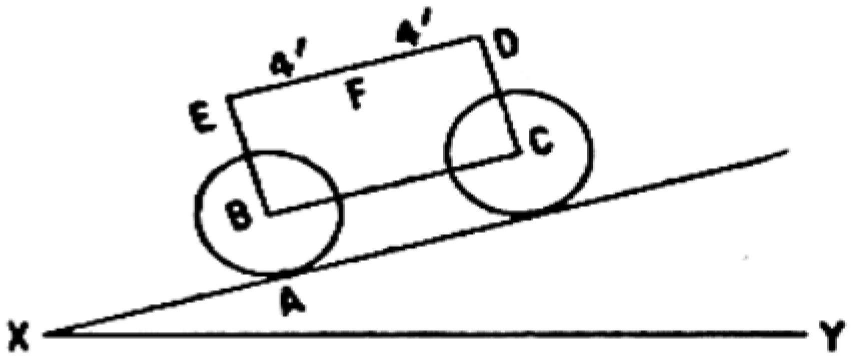

FIG. 56.

2. A truck rests on a slope of 30° to the horizontal XY (Fig. 56). If EB = 3 ft. = DC, BC = 8 ft. = ED, XA = 5 ft., and the wheels have radius 1 ft. and centres B, C, find the heights, above XY, of B, C, D, F (mid point of ED).

3. ABCD is a quadrilateral with sides AB = 8 in., BC = 5 in., CD = 3 in., DA = 6 in. and angle ABC = 60°. Calculate angle CDA.

[Find AC by means of the cosine formula.]

4. ABCD is a quadrilateral of jointed rods;

AB = 5 in. = AD, CB = 12 in. = CD, AC = 13 in.

What is angle ABC? Find BD. What change is caused in BD if AC becomes 16 in.? [Calculate angle BAC.]

IIId

1. An aircraft flies 100 miles in a direction N.70°E. and then 50 miles in a direction S.50°W. How far is it then (i) north, (ii) east of its starting point?

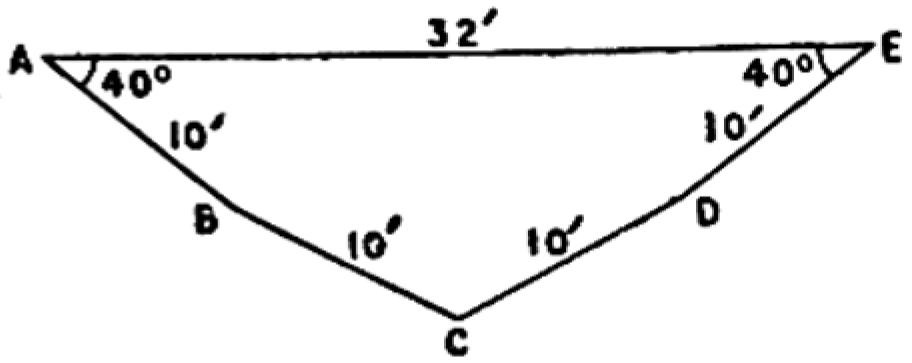

FIG. 57.

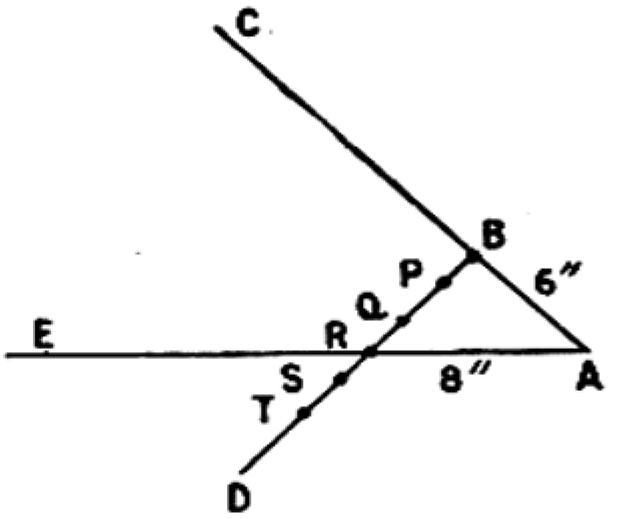

FIG. 58.

2. In Fig. 57, AE is horizontal and 32 ft. long. AB = BC = CD = DE = 10 ft., and angles A and E are

each 40°. Find the inclination of BC to the horizontal, and the depth of C below AE.

3. A triangle ABC has AB = 31 in., BC = 24 in., CA = 11 in. BC is produced to D where CD = 35 in. Find the distance DA.

4. ABC (Fig. 58) is a window which can be secured by an arm BD in which are holes P, Q, R, S, T, 2 in. apart. The holes engage with a peg 8 in. from A in the window frame AE. BP = 2 in., AB = 6 in. Calculate the angle BAE when the holes P, Q, R, S, T are on the peg in turn.

5. Two nails, P, Q, at the same level are 2 ft. apart. A long string hangs over the nails and carries equal weights A, B at its ends. The middle point M of the string is pulled down until A and B have each risen 3 ft. What is the angle between the parts PM, QM of the string ?

CHAPTER IV

THE TANGENT AND COTANGENT

Tangent of Acute Angle

IN Fig. 59 it is required to find AB, the height of a tower, being given that at C, a point 150 ft. from its foot, $\angle ACB = 36^\circ 18'$ (the angle of elevation). What is needed is the ratio of AB, the side opposite the angle, to BC the side adjacent to it. This, for a reason suggested on p. 51, is called the *tangent* of C. In Fig. 60, in which $\angle R$ is a right angle, the tangent of θ or, for short, $\tan\theta = \frac{\text{opp.}}{\text{adj.}}$

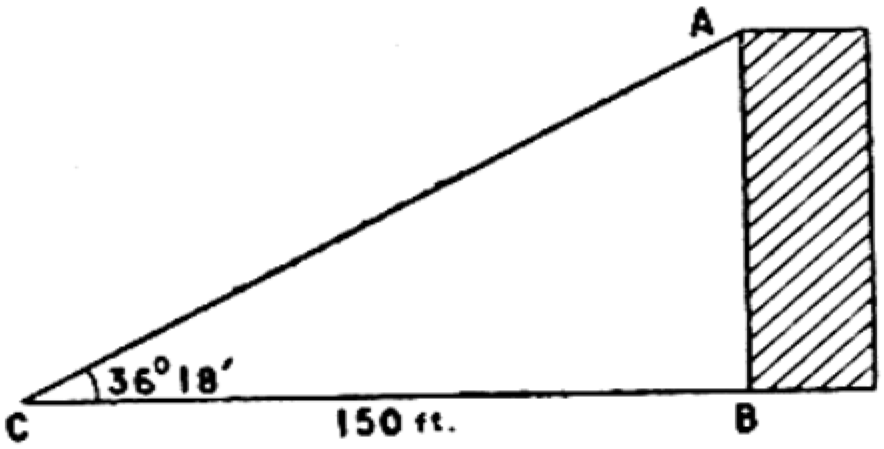

FIG. 59.

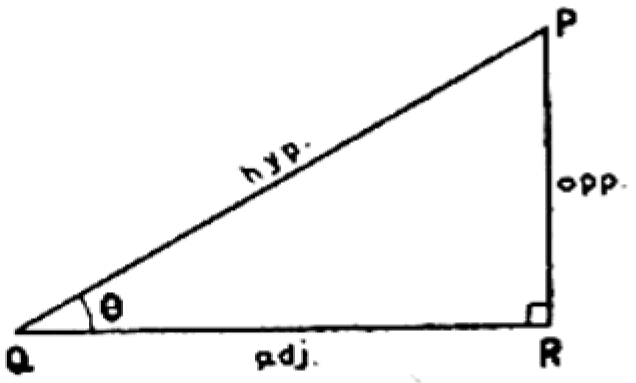

FIG. 60.

The values of this ratio are found in the table of the natural tangents.

So in Fig. 59, $\frac{AB}{150} = \tan 36^\circ 18' = 0{\cdot}7346.$

$\therefore$ AB $= 150 \times 0{\cdot}7346 = 110{\cdot}2.$

The tower is just over 110 ft. high.

$$\begin{array}{r}73{\cdot}46\\36{\cdot}73\\\hline 110{\cdot}19\end{array}$$

Tangent $= \frac{\textbf{sine}}{\textbf{cosine}}$. In Fig. 59, $\frac{AB}{BC} = \frac{\sin C}{\sin A}$, by the sine rule. But $\angle A$ = complement of $\angle C$. $\therefore \sin A = \cos C$.

Thus
$$\frac{AB}{BC} = \tan C = \frac{\sin C}{\cos C}.$$

It is quicker to use the tangent for the above problem than to use either

$$\frac{AB}{150} = \frac{\sin 36^\circ 18'}{\sin 53^\circ 42'} \quad\text{or}\quad \frac{AB}{150} = \frac{\sin 36^\circ 18'}{\cos 36^\circ 18'}.$$

Logarithmic tangents (*i.e.*, logarithms of tangents) may be used ; see Ex. 2.

Examples—23

1. In Fig. 59 find the height of the tower if BC = 200 ft. and ∠C = 29°42′.

2. From a point 340 ft. from the foot of a tower, the angle of elevation of its top is 19°43′. Find the height of the tower.

[Sol'n. The height = 340 tan 19°43′ Logs 2·5315
= 121·8 ft. $\overline{1}$·5543
2·0858.]

3. As in Ex. 2, find the height of
(i) Distance is 450 ft., angle of elevation 16°51′.
(ii) Distance is 227 ft., angle of elevation 58°10′.

4. In a triangle ABC right-angled at A it is given that $b = 7{\cdot}2$ cm., $c = 4{\cdot}3$ cm., find B and C.

	By division	or by logs
[Sol'n. $\tan C = \dfrac{c}{b}$	72)43·0/·5972	0·6335
	36·0	0·8573
$= \dfrac{4{\cdot}3}{7{\cdot}2}$	700	$\overline{1}$·7762
	648	
C = 30°51′	520	
B = 59°9′.]	504	
	160	

5. Find the other angles of the triangles in which
(i) A = 90°, $b = 4{\cdot}7$ cm., $c = 9{\cdot}3$ cm.
(ii) B = 90°, $c = 63$ yd., $a = 48$ yd.

6. A tower is 56 ft. high. What is its angle of elevation at a distance 120 ft. from its foot ?

7. A boat is 85 yd. from the foot of a cliff 90 ft. high. What is its angle of depression from the top of the cliff ?

8. (i) A vertical pole 9 ft. high casts a shadow 14 ft. long. What is the "altitude" of the sun ?

(ii) If the sun is at an elevation of 56°36′ and the shadow of a vertical pole is 8 ft. 6 in. long, what is the height of the pole ?

9. An aeroplane is at a height of 2000 ft. (i) What is the angle of depression of an object on the ground whose distance horizontally is half a mile ? (ii) If from the same plane you look down at an angle of depression 15°, how far off horizontally is the object ?

10. ABC is an isosceles triangle with AB, AC equal; AD is the perpendicular from A to BC.

(i) If $\angle B = 62°$, BC = 8 in., find AD.

(ii) If $\angle A = 102°40'$, BC = 12 cm., find AD.

11. The vertical angle of a cone is 76°. If the diameter of its base is 14 cm., find its height.

[Examples 1 to 5, pp. 18, 19, will afford further practice.]

The Tangent by Measurement

12. Draw $\triangle ABC$ having AC = 5 in., $\angle A = 90°$, $\angle C = 36°$. Measure AB and hence find tan 36°. Compare with the tables.

13. In a triangle the sides containing the right angle are 9 cm. and 7 cm. Draw the triangle accurately and measure its smallest angle. Compare this with the result of calculating the angle.

The Tangent Graph

If the radius CA of a circle is taken as unit length, and a radius CP is produced to meet at T the tangent to the circle at A, then

$$\tan \angle ACT = \frac{AT}{AC} = \frac{AT}{1} = AT$$

and the *tangent of the angle* is shown on the *tangent to the circle.*

If AZ, AY, . . . AU, AT are the lengths for tan 10°, tan 20°, . . . tan 60°, tan 70°, these lengths may be transferred, as shown, to enable a graph to be drawn. The construction shows clearly that however tall the paper the graph will leave it before 90° is reached.

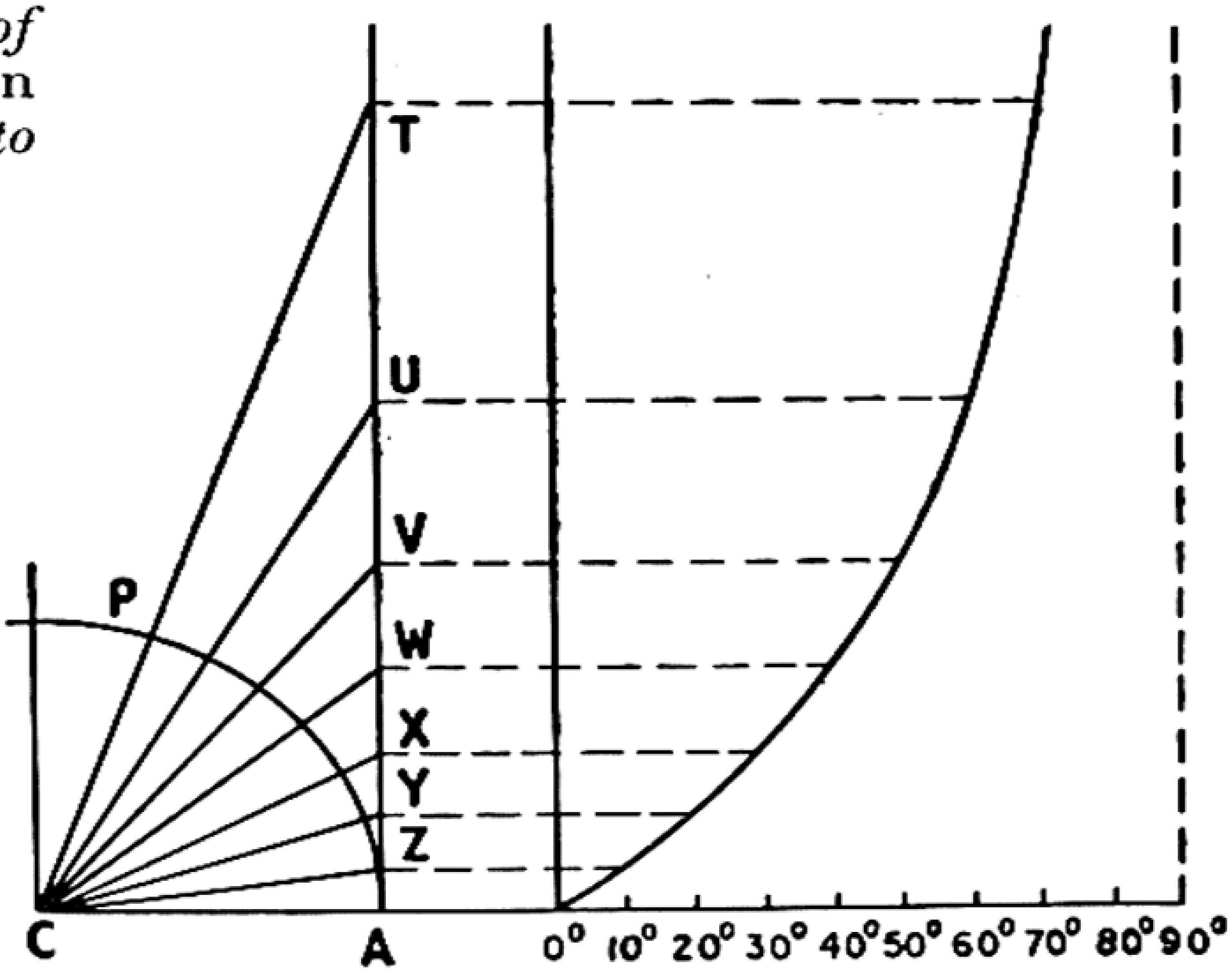

FIG. 61.

Examples—24

1. Draw the above graph, taking 5 cm. as the radius of the circle. On the same figure, by transferring the heights of the points P where CT, etc., cut the circle, draw the graph of the sine. Note that the sine is always less than the tangent, though up to 10° there is little difference.

2. From the graph state (i) which is the greater, tan 30° — tan 20° or tan 60° — tan 50° ? (ii) which is the greater, tan 30° — sin 30° or tan 50° — sin 50° ?

3. If the length of CA in Fig. 61 is one inch, what would be the distance AT when ∠ACP is (i) 80°, (ii) 89°, (iii) 89°54′ ?

The Tangent as Gradient

Just as the cosine is important as giving the *projection* of a line, so the tangent is important as giving its *gradient.*

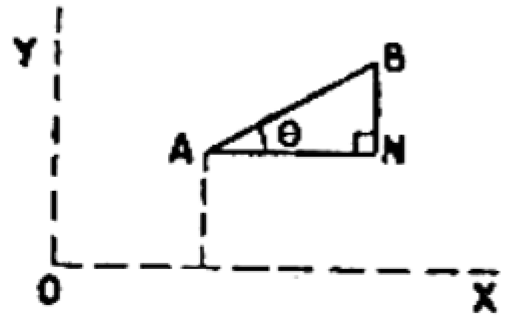

FIG. 62.

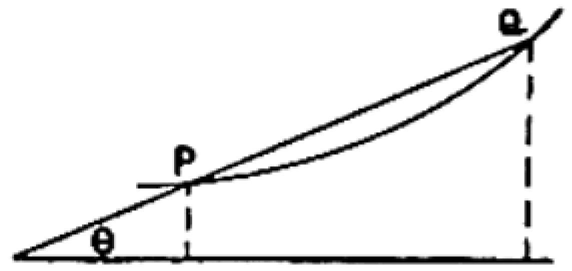

FIG. 63.

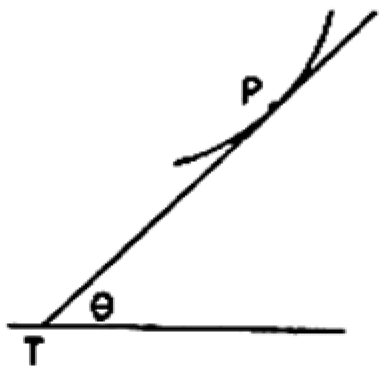

FIG. 64.

In the above figures the x-axis or horizontal line is drawn across the page.

In Fig. 62, AB slopes upward ; the angle of slope is ∠NAB or θ.

The gradient of AB is the ratio of the vertical distance (or rise) NB to the horizontal distance AN, *i.e.*, BN/AN or tan θ.

In Fig. 63, PQ is a chord of a curve and tan θ its gradient.

In Fig. 64, PT is the tangent at P to a curve ; tan θ, the gradient of the *line* TP, is the gradient of the *curve* at P.

Examples—25

1. A hill rises 37 ft. in a horizontal distance of 100 ft. What is the gradient ? Find its angle of slope.

2. What is the gradient of the line joining
(i) (2, 4) to (9, 15) ; (ii) (0, 0) to (4, 5) ; (iii) (—2, 0) to (5, 7)?

3. What are the gradients of the lines CX, CY, CZ in Fig. 61 ?

4. In Fig. 64, above, the curve is part of a circle and the *radius* to P slopes down at 62° to the horizontal. What is the gradient of the curve at P ?

5. Sketch a curve and a chord like Fig. 63, above, and show in your sketch how to divide the curve between P and Q into two parts, in one of which the gradient of the curve is less than that of PQ and in the other greater than that of PQ.

6. The projection of AB on the horizontal is ·8 AB. What is the gradient of AB ?

7. In Fig. 63, opposite, $\sin \theta = \cdot 55$; what is the gradient of PQ ?

8. Mr. Smith walks uphill for 100 yds. at a gradient of 1 in 8 (*i.e.*, $\frac{1}{8}$) and then downhill for 50 yds. at a gradient of 1 in 4. Is he above or below his starting-point ?

9. If, in Fig. 62 above, A is the point (x_1, y_1) and B the point (x_2, y_2), what is the gradient of AB ?

10. Engineers on roads and railways take NB/AB (Fig. 62) as the gradient, *i.e.*, rise/length of track, giving $\sin \theta$. What is the answer to Ex. 8 if this method of reckoning gradients is used ?

11. What is the difference, in angle of slope, between the engineer's and the mathematician's gradients in the case of (i) a steep road rising 1 in 5, (ii) a steep railway rising 1 in 50 ?

Distance and Dip of Horizon

If a tangent is drawn to a sphere of radius R from a height z above it, the product theorem gives for x, the length of the tangent

$$x^2 = z(2R + z).$$

If z is small compared with R, as when R is the radius of the earth and z the height of an accessible point above the earth's surface, this gives

$$x^2 \doteqdot 2zR$$

from which to find the distance of the horizon.

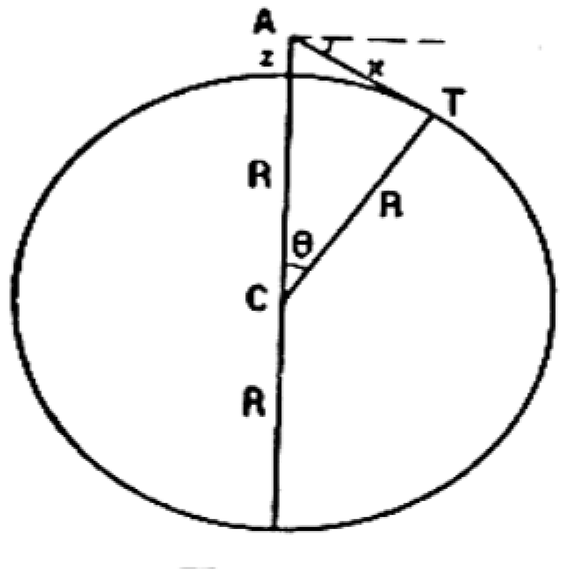

FIG. 65.

Also if θ is the "dip" or angle of depression of the horizon, Fig. 65 shows that

$$\tan\theta = \frac{x}{R} = \sqrt{\frac{x^2}{R^2}} = \sqrt{\frac{2zR}{R^2}} = \sqrt{\frac{2z}{R}}.$$

The first relation gives a remarkable rule, which everyone should know;

From a height h feet above the sea,
the distance of the horizon is $\sqrt{(1\frac{1}{2}h)}$ *miles.*

This is got by taking $R = 3960$ miles (instead of the usual 4000 miles) and h feet $= h/5280$ mi., so that

$$x = \sqrt{(2.3960\,h/5280)} = \sqrt{(1\tfrac{1}{2}h)}.$$

Thus at 600 ft. up you can see $\sqrt{900} = 30$ miles.

From Snowdon, 3560 ft., the distance of the horizon would be $\sqrt{5340} \bumpeq 73$ mi.

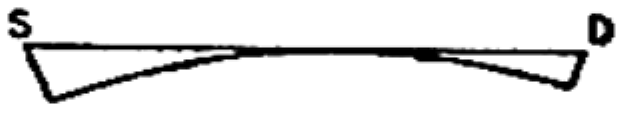

Fig. 66.

This would not quite reach the Irish coast, but it is easy to see the Dublin mountains, some 10 miles inland and 2000 ft. high, for $\sqrt{3000} \bumpeq 54$ mi. can be added. (See Fig. 66.)

Examples—26

1. Find the dip of the horizon from the top of Snowdon. Show that if θ is the dip from a height of h ft., $\tan\theta \bumpeq 0{\cdot}0003\sqrt{h}$.

2. Make a table showing roughly the distance and dip of the horizon for heights 10,000 ft., 20,000 ft., 30,000 ft., 40,000 ft.

3. What area of sea is visible from the masthead of a ship if it is 80 ft. above the water?

4. From the top of a mountain the distance of the visible horizon is 42 miles; what is the dip of the horizon? [See Fig. 65.]

5. Two ships have masts up which it is possible for a man to climb to a height of 200 ft. above the water. What is the greatest distance at which a man on the mast of one is visible to a man on the mast of the other?

6. Prove the formula $x^2 = z(2R + z)$ by applying Pythagoras' theorem to the triangle CAT (Fig. 65).

The Cotangent of an Acute Angle

Just as the sine of the complement of θ is called the cosine of θ, so the *tangent* of the complement of θ is called the *cotangent* of θ.

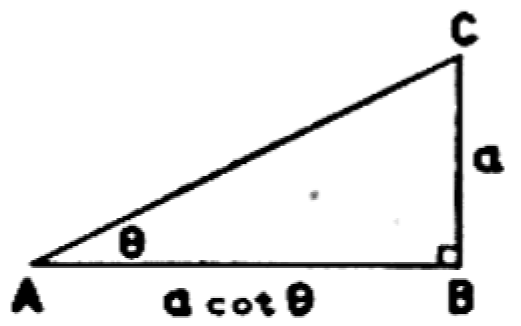

FIG. 67.

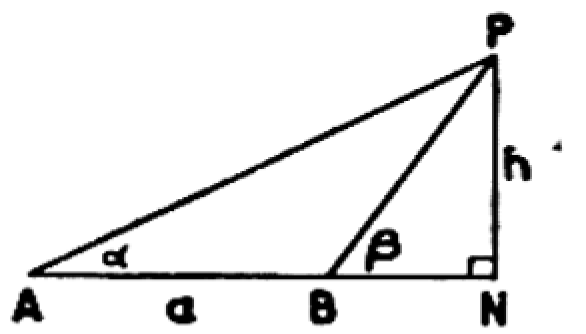

FIG. 68.

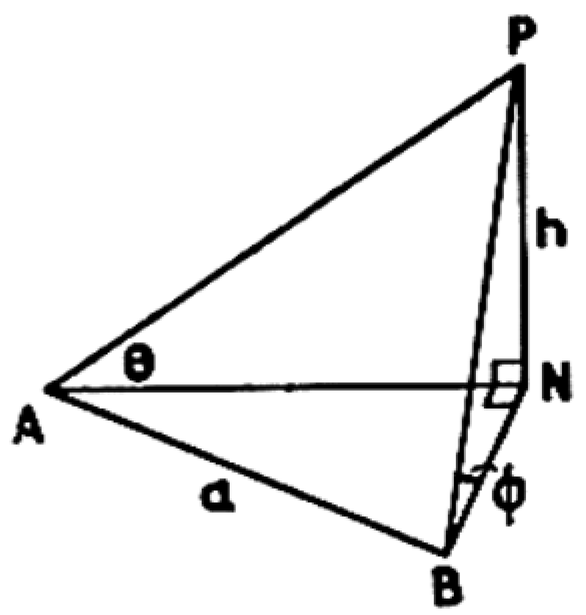

FIG. 69.

In Fig. 67, $\frac{AB}{BC}$ is the cotangent of A, or, for short, cot A. For $\frac{AB}{BC}$ is tan C, and $\angle$C is the complement of $\angle$A.

Notice that $\cot A = \frac{1}{\tan A}$, for $\tan A = \frac{BC}{AB}$; also that $\cot A = \frac{\cos A}{\sin A}$.

Either of these statements may be taken as defining cot A. One of the chief uses of the cotangent is to express horizontal distances in terms of vertical distances, or to "get down from the air to the ground." It is more convenient to multiply by a cotangent than to divide by a tangent.

If, in Fig. 68, PN is a flagstaff of height h observed from A and B, then $AN = h \cot \alpha$ and $BN = h \cot \beta$, so

$$a = h(\cot \alpha - \cot \beta).$$

Hence, if AB is known, h can be found, or *vice versa*.

Similarly, if in Fig. 69 a tower PN is observed from points A,B on the level ground, with $\angle$ANB a right angle, then $AN = h \cot \theta$ and $BN = h \cot \phi$, $\therefore\ a^2 = h^2 \cot^2 \theta + h^2 \cot^2 \phi$. whence h can be found if a is known or measured.

Examples—27

1. $\triangle$ABC has a right angle at B. If $a = 120$ ft. and $c = 147{\cdot}3$ ft., which is the easier way of finding the angle at A (i) using $\tan A = \frac{120}{147{\cdot}3}$ or (ii) using $\cot A = \frac{147{\cdot}3}{120}$? Find $\angle$A the easier way.

2. An aeroplane flying at a height of 1000 ft. has an elevation of 23°30′ from A; how far is it from A horizontally?

3. In Fig. 68, if
   (i) $h = 100$ ft., $\alpha = 25°$, $\beta = 52°$, find a.
   (ii) $a = 150$ ft., $\alpha = 29°30'$, $\beta = 63°$, find h.

4. At a point 70 ft. from its foot the angle of elevation of the top of a tower is 45°. How much further away must one go, so that the angle of elevation may be 20°?

5. In Fig. 69, if $h = 120$ ft., $\theta = 26° = \phi$, and ANB is a right angle, find a.

6. In Fig. 69, P represents the top of a hill. The elevations of P from points A (due W.) and B (due S.) are 23°45′ and 34°56′. If the distance AB is 2230 ft., find the height PN.

7. Draw the graph of $\cot\theta$ from $\theta = 30°$ to $\theta = 90°$.

8. In Fig. 68, if $\beta = 2\alpha$ prove that $BN < AB$. Hence show that $\cot 2\alpha < \frac{1}{2}\cot\alpha$.

Examples—28

Problems in Three Dimensions

1. A hexagonal pyramid whose base-edges are 6 ft. long has a vertical height of 10 ft. Find the angle between a slant edge and the base.

2. A symmetrical pyramid VABCD has a rectangular horizontal base ABCD, with AB = 16 ft. and BC = 12 ft., and its vertical height is 20 ft. Calculate the inclinations to the horizontal of (i) the faces VAB and VBC, (ii) the slant edge VA.

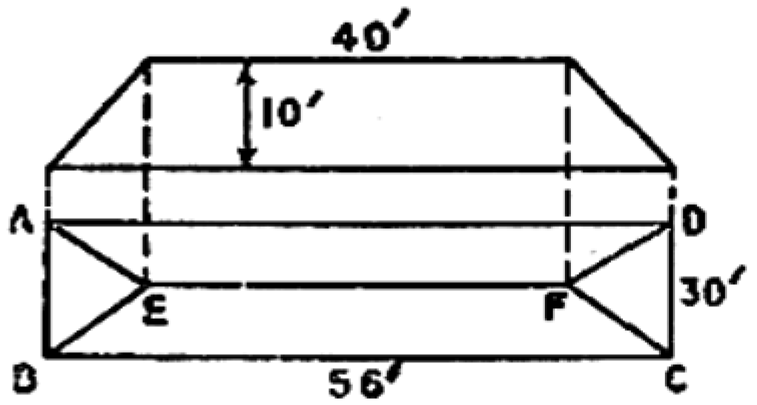

Fig. 70.

3. Fig. 70 (not to scale) shows the side-elevation and plan of a roof. Calculate the angles made with the horizontal by the faces ABE and AEFD.

4. At one corner C of a rectangular field ABCD is a tall vertical pole PC. The angles of elevation of P from B and D are 30° and 45° respectively. What is the elevation from A?

[Take PC = 100 ft. and calculate in turn CD, CB and CA.]

5. The top of a hill P is observed from two points A and B, A being due S. and B due W. of the top. The point P is 600 ft. above the level of A and B, and the elevations of P from A and B are 26° and 29°. Calculate AB.

6. A right triangular prism $ABCA'B'C'$ has $AC = 6''$, $BC = 7''$, angle $ACB = 60°$, and $AA' = BB' = CC' = 5''$. $AX = B'Y = 1''$. Find AB and the volume of the prism. (Fig. 71.)

Find also the inclination of XY to AB, using the cotangent. [The edges AA′, etc., are at right angles to the base.]

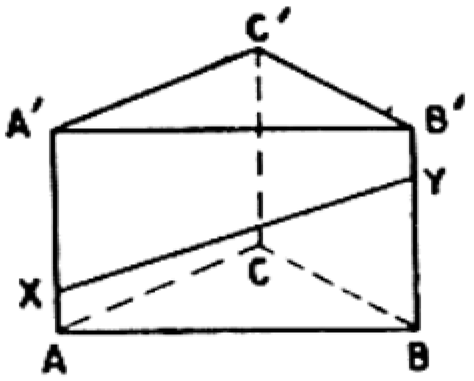

FIG. 71.

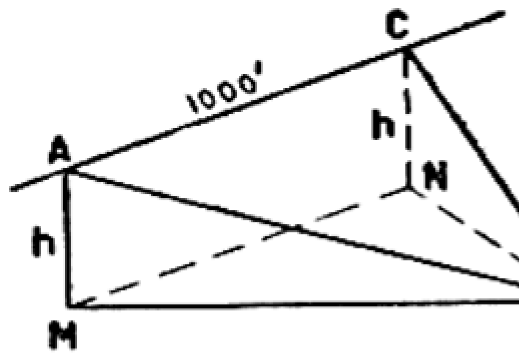

FIG. 72.

7. In Fig. 70, find the angle between EC and the horizontal, using the cotangent.

8. A horizontal cliff-edge runs north and south (Fig. 72). A boat B is due E. of a point A on the cliff edge and its angle of depression from A is 35°. From a point C, 1000 ft. further north along the cliff, the angle of depression of the boat is 25°. Find the height of the cliff. [Express MB, NB in terms of h; $\angle BMN = 90°$.]

9. The roof of a building is shown in Fig. 73. AB = 40 ft., BC = 100 ft., and the flat top $A'B'C'D'$ is 10 ft. above the level of A, B, C. All four sloping faces are inclined at 55° to the horizontal. Calculate (i) the dimensions of the flat top, (ii) the angle between BB′ and the horizontal.

10. If, in Fig. 73, the face ABB′A slopes at 45° to the horizontal, and $AB = 40'$, $BC = 100'$, $A'B' = 30'$, $B'C' = 82'$, find the height of $A'B'C'D'$ above ABCD and the slope of the face BCC′B′.

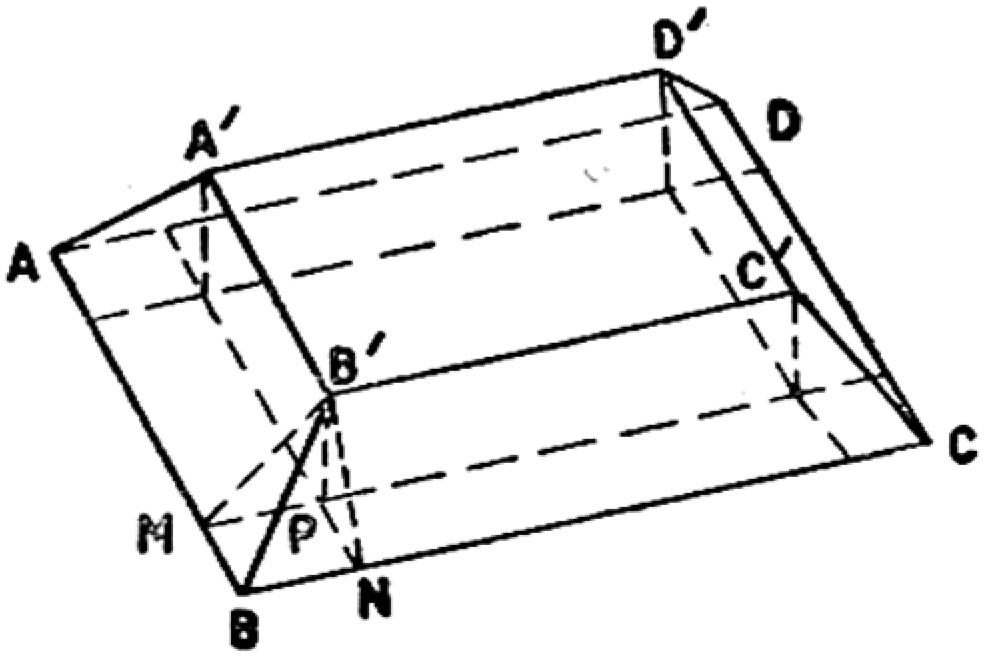

FIG. 73.

Tangents of Obtuse Angles

The tangent of an acute angle θ has been shown to be $\frac{\sin\theta}{\cos\theta}$.

This is taken as the definition in the case of an obtuse angle.

So when θ is obtuse $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Since $\cos\theta$ is negative if θ is obtuse, it follows that $\tan\theta$ is negative also and that

$$\tan\theta = \frac{\sin(180^\circ - \theta)}{-\cos(180^\circ - \theta)} = -\tan(180^\circ - \theta).$$

Thus $\tan 141^\circ = -\tan 39^\circ$, $\tan 160^\circ = -\tan 20^\circ$, and so on.

Cotangents of Obtuse Angles

For an acute angle $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

For an obtuse angle this is taken as the definition of $\cot\theta$.

It follows that if θ is obtuse $\cot\theta = \frac{1}{\tan\theta} = -\cot(180^\circ - \theta)$.

Thus $\cot 141^\circ = -\cot 39^\circ$, $\cot 160^\circ 20' = -\cot 19^\circ 40'$, and so on.

Examples—29

1. Find from the tables $\tan 100^\circ 25'$ and $\tan 156^\circ 37'$; also find the cotangents.

2. The cosine of an angle is $-0{\cdot}7$; find its tangent and cotangent.

3. The sine of an obtuse angle is $0{\cdot}4$; find its tangent and cotangent.

4. If $\tan\theta^\circ + \tan 21^\circ 5' = 0$, find $\sin\theta$, $\cos\theta$ and $\cot\theta$.

5. AB, BC are chords of a circle of length 4 in. and 3 in. respectively. B lies on the circle between A and C and the radius of the circle is 3 in. Find tan ABC.

6. CB, CD are tangents from C to a circle of radius 4 in. and centre A. If CB = CD = 5 in., find tan BCD and tan BAD [first find these angles].

7. ABC is an acute angled triangle and the perpendicular from A to BC is of length h_1; prove that

$$BC = h_1(\cot B + \cot C).$$

Is this true if B is obtuse ?

8. If $\angle A$ is a right angle, prove that $\triangle ABC = \frac{1}{2}c^2 \cot C$.

9. What is the locus of a point from which the elevation of the top of a lighthouse 150 ft. high is 27°, the point being at the level of the foot of the lighthouse ?

10. A bust of Napoleon is on a bracket on a wall 8 ft. from the ground. A bust of Wellington is on a chimney piece exactly opposite across a room 16 ft. wide at a height of 4 ft. 6 in. from the ground. What is the angle of depression of Wellington from Napoleon ?

11. Draw on the same diagram the graphs of $\tan \theta$ and $\cot \theta$ for values of θ from 20° to 70°. Where do they cross ?

12. Draw on same diagram the graphs of $\tan \theta$ and $\cot \theta$ for values of θ between 120° and 150°. Which of these could you continue as far as 180° ?

Solution of Triangles by Tangent and Cotangent

In solving triangles in the S.A.S. case, either of the following formulæ can be used :—

I. $\dfrac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \dfrac{a - b}{a + b}$. The *tangent* formula.

II. $\cot B + \cot C = \dfrac{a}{b \sin C}$ or $\dfrac{a}{c \sin B}$. The *cotangent* formula.

Of these formulæ the first is more commonly used ; it is often called " Napier's Analogy," the word " analogy " being used in its old sense of " proportion."

The second formula is easily proved. It is equivalent to dividing the triangle into two right-angled triangles and solving these. It is not so symmetrical as the tangent formula but it is not hard to remember if it is noticed that the two letters on the left correspond to those in the *denominator* on the right, the numerator being the third letter.

Examples—30

The Cotangent Formula

1. In $\triangle ABC$ draw $AD \perp BC$. Let $AD = h$. Use $BD = h \cot B$, $DC = h \cot C$, $h = b \sin C$, to prove that

$$\cot B + \cot C = \frac{a}{b \sin C}.$$

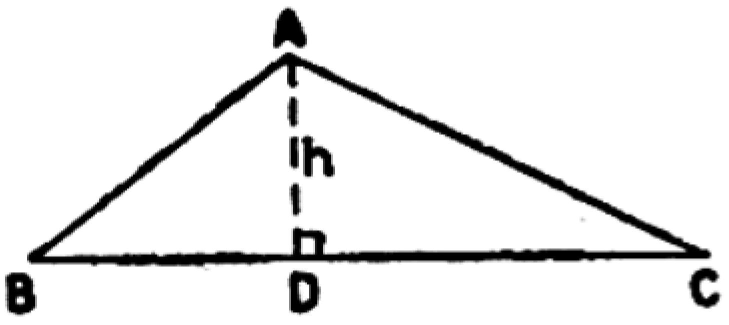

FIG. 74.

2. Use the cotangent formula to find B if

(i) $a = 7{\cdot}3$ in., $b = 4{\cdot}2$ in., $C = 51°14'$.
(ii) $a = 5{\cdot}4$ in., $b = 5{\cdot}1$ in., $C = 70°30'$.

$$\begin{array}{lll} \text{[Sol'n of (i)}\ \dfrac{a}{b\sin C} = \dfrac{7{\cdot}3}{4{\cdot}2\sin 51^\circ 14'} & \begin{array}{l} 0{\cdot}8633 \\ \underline{0{\cdot}5151} \\ 0{\cdot}3482 \end{array} & \begin{array}{l} 0{\cdot}6232 \\ \underline{\bar{1}{\cdot}8919} \\ 0{\cdot}5151 \end{array} \\ \qquad\qquad\qquad = 2{\cdot}229 & & \end{array}$$

$\cot C = 0{\cdot}8031$ (but note that the last figure is wasted)

$\therefore \cot B = 1{\cdot}426$ and $B = 35°2'$.]

3. Prove, as in Ex. 1, that even if B is obtuse,

$$\cot A + \cot B = c/a \sin B.$$

Use this to find A if $a = 76$ yd., $c = 48$ yd., $B = 125°16'$.

[The cotangent formula is useful in surveying, in finding the heights of an inaccessible object.]

4. Suppose BC in Fig. 74 is a base line 1000 yd. long, and the angles of elevation of A from B and C are 27°40′ and 22°32′. Then $h\,(\cot 27°40' + \cot 22°32') = 1000$, from which h is found. Complete the solution.

5. The angle of elevation of the top of a tower from a point B is 18°47′. From a point 66 yd. nearer the tower the elevation is 28°47′. Find the height of the tower.

6. A weight is suspended from points A, B (at the same level) by two strings 6·4 ft. and 8·3 ft. long, the angle between them being 106°38′. Calculate the depth of the weight below AB.

Napier's Analogy

7. In Fig. 75 the circle centre C, radius CA ($= b$) cuts BC at P and Q.

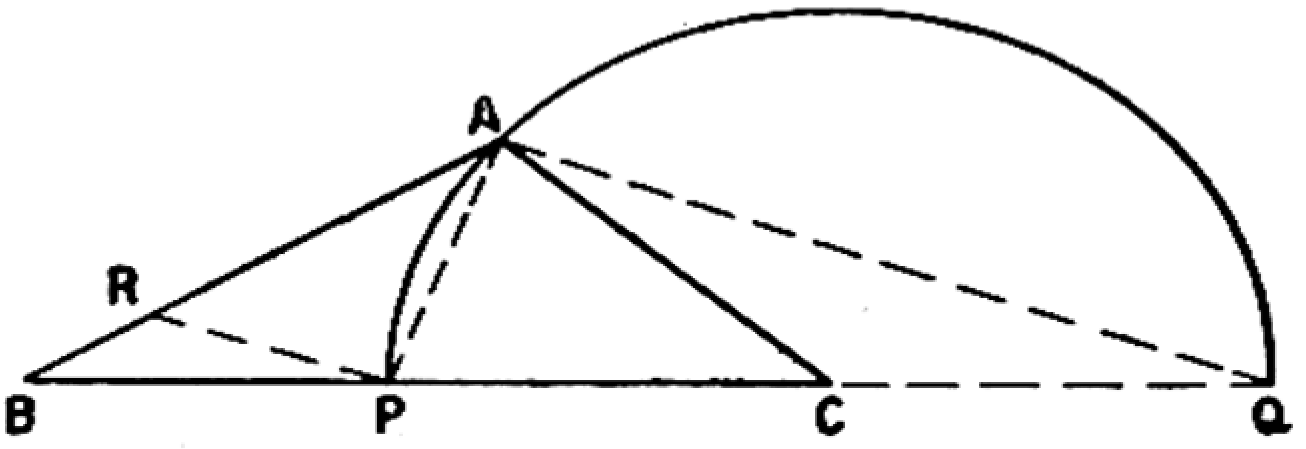

FIG. 75.

Prove (i) $\angle APC = \frac{1}{2}(A + B)$

$\angle PAR = \frac{1}{2}(A - B)$

(ii) $BQ = a + b$, $BP = a - b$.

And if $PR \parallel QA$, so that $AP \perp PR$,

(iii) $\tan \frac{1}{2}(A + B) = \dfrac{AQ}{AP}$, $\tan \frac{1}{2}(A - B) = \dfrac{RP}{AP}$

(iv) $$\frac{RP}{AQ} = \frac{BP}{BQ}$$

and so $$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{a - b}{a + b}.$$

8. Solve questions 2 (i) and 2 (ii) by using Napier's Analogy.

[Sol'n of (i) $C = 51°14'$, $\therefore A + B = 128°46'$

$\therefore \frac{1}{2}(A + B) = 64°23'$

$a - b = 7\cdot3 - 4\cdot2 = 3\cdot1$, $a + b = 11\cdot5$

(Check : $a - b + a + b = 2a$, $3\cdot1 + 11\cdot5 = 14\cdot6$.)

$$\therefore \tan \tfrac{1}{2}(A - B) = \frac{3\cdot1 \tan 64°23'}{11\cdot5}$$

$\therefore \frac{1}{2}(A - B) = 29°21'$

But $\frac{1}{2}(A + B) = 64°23'$

$\therefore B = 35°2'$.]

$$\begin{array}{r} 0\cdot4914 \\ \underline{0\cdot3193} \\ 0.8107 \\ \underline{1\cdot0607} \\ \bar{1}\cdot7500 \end{array}$$

9. Re-draw Fig. 75 making C obtuse, and show that this makes no difference to the proof.

10. (i) Show that the formula

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} . \cot \tfrac{1}{2}C$$

is equivalent to the formula in Ex. 7.

10. (ii) B is 12 miles due N. of C, and A is 8 miles N.70°E. from C. Prove that $\tan \frac{1}{2}(A - B) = \frac{1}{5} \cot 35°$, and find the bearing of A from B.

11. Write down the version of Napier's formula to be used if c, a and B are given, and hence solve the triangle for which $c = 216$ yd., $a = 123$ yd., $B = 25°42'$.

12. Find the angles A and B in a triangle if $a = 142$ ft., $b = 327$ ft., $C = 64°$. [Here $b > a$, so we use the formula of Ex. 7 with A and B interchanged. In the proof of the formula it was assumed that $b < a$.]

13. Two sides of a triangle are in the ratio of $5:7$, and the angle between them is $75°$. Calculate the other two angles.

[Take the sides as $5x$ and $7x$ and use Napier's formula.]

14. If $a = 8$, $b = 6$ and $C = 30°$, calculate A by three methods: (i) the cotangent formula, (ii) Napier's formula, (iii) the cosine formula followed by the sine formula.

Examples—31: Miscellaneous

1. Use Fig. 76 to prove Napier's formula, by going through the following steps:—

(i) $\angle CAY = \frac{1}{2}(B + C) = \angle BAO$.
(ii) $\angle BCX = \frac{1}{2}(B - C)$.
(iii) $CX = YA + AO = (b + c) \cos \frac{1}{2}(B + C)$.
(iv) $BX = YC - OB = (b - c) \sin \frac{1}{2}(B + C)$.
(v) $\tan \frac{1}{2}(B - C) = BX/XC$.

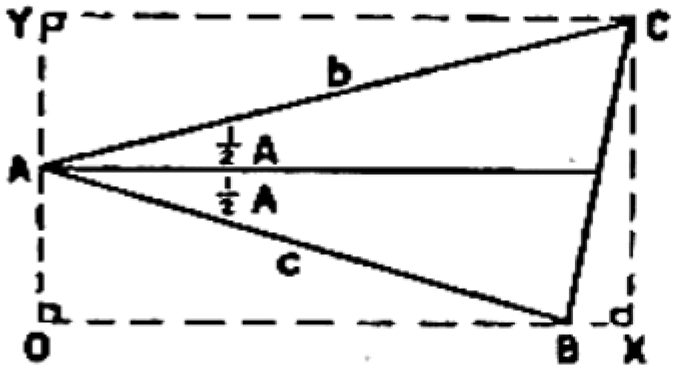

FIG. 76.

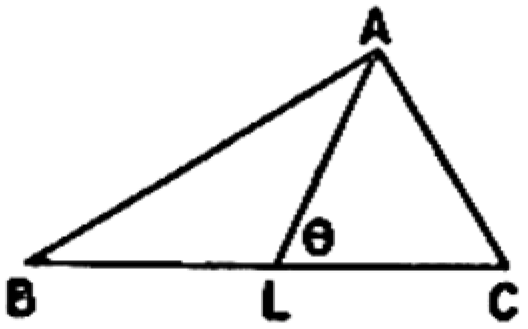

FIG. 77.

2. In Fig. 77 AL is a median of $\triangle ABC$, *i.e.*, $BL = LC$. Prove that $2 \cot \theta = \cot B - \cot C$.

[Apply the cotangent method to $\triangle$s ABL, ACL by drawing $AD \perp BC$.]

3. If, in Fig. 77, $BL = \frac{1}{3}BC$, prove that

$$\cot \theta = \tfrac{2}{3} \cot B - \tfrac{1}{3} \cot C.$$

4. If the median AL of a triangle ABC (Fig. 77) makes $\angle\beta$ with AB and $\angle\gamma$ with AC, and $\angle ALC = \theta$, prove that

$$\cot \theta = \tfrac{1}{2}(\cot \beta - \cot \gamma).$$

[Draw $CF \perp AL$ and $BE \perp AL$ produced. Prove $CF = BE$ ($= h$, say) and $AL = h \cot \beta - h \cot \theta = h \cot \gamma + h \cot \theta$.]

5. If r is the radius of the inscribed circle of the $\triangle ABC$, prove that

$$\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C = \frac{a}{r}.$$

[Apply the cotangent method to $\triangle IBC$.]

6. Write down the two other equations, similar to the one in Ex. 5, and use them to find a, b, c when $r = 1''$ and $A = 50°$, $B = 60°$, $C = 70°$.

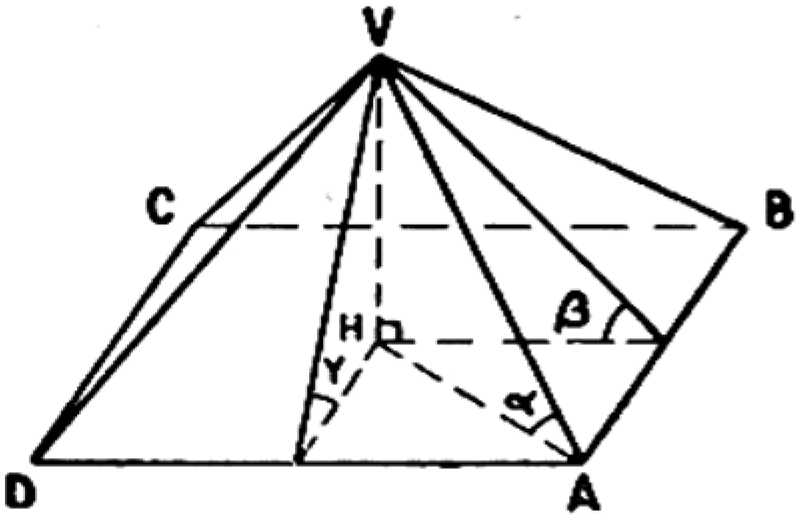

FIG. 78.

7. A pyramid (Fig. 78) has a rectangular base ABCD and vertex V. If α, β, γ are the inclination to the horizontal base of the edge VA, the face VAB and the face VAD, prove that

$$\cot^2 \alpha = \cot^2 \beta + \cot^2 \gamma.$$

REVISION EXERCISES

IVA

1. A bicycle shed is 8 ft. from front to back. The roof is 6 ft. above ground at the front and 8 ft. at the back. What is the angle of slope of the roof ?

2. The angle of elevation of the top of a tower is 48° from a point 100 ft. from its base. What is the angle of elevation of a brick half-way up the tower, from the same point ?

3. The tangents to a circle, of diameter 6 in., from a point 8 in. from its centre are drawn. What is the angle between them ?

4. A hill-top is 600 ft. above sea-level. A sailor knows that if the angle of elevation of this hill-top is more than 23° he is in danger of striking a submerged rock. How far clear of the rock is he if he keeps the angle at 21° ? (In such a case 23° is called a " danger angle ").

IVB

1. A conical tent is 10 ft. high and the diameter of the base is 8 ft. What is the angle between the canvas and the tent-pole ?

2. On a map whose scale is six inches to the mile a straight track from A to B measures 0·74 in. A is on the 400-ft. contour and B on the 600-ft. contour. What is the average angle which the track makes with the horizontal ?

3. ABCD is a vertical section of a roof, with BC horizontal. AB is 18 ft. and has a slope of 47°, CD is 20 ft. and has a slope of 57°, and BC is 30 ft. What is the slope of AD to the horizontal? [Find the heights AM, DN of A and D above BC, and this distance MN.]

4. A small weight W is hung from a roof by two ropes AW, BW whose inclinations to the horizontal are 38° and 48°. The points of attachment A, B of the ropes to the roof are 12 ft. apart and at the same level. How far below the roof is the weight? [Show that d (cot 38° + cot 48°) = 12.]

IVc

1. To find the width of a river, a man drives a stake into the bank at S opposite a tree T on the other side. He then walks 100 yards along the bank to R so that RST is a right angle, and finds that the angle SRT is 49° (using a sextant). What is the width of the river?

2. A pyramid whose height is 17 ft. stands on a square base whose side is 10 ft. What are the angles between (i) a slant face and the base, (ii) a slant edge and the base?

3. The altitude of the sun in London at mid-day varies from 15° in December to 62° in June. The length of the shadow of a certain flagstaff (at midday) shortens by 64 ft. during this period. How high is the flagstaff?

[Show that h (cot 15° — cot 62°) = 64.]

4. A piece of ground sloping at 10° to the horizontal is represented in section by AB in Fig. 79. It is to be levelled by removing the earth from MNB and depositing it at MLA. For safety the slopes of AL and NB must not be more than 40°. Find the levelled width LN if AM = 50 ft. = MB.

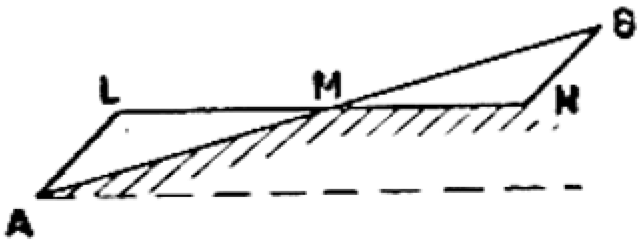

FIG. 79.

[Apply the sine formula to △ALM.]

IVD

1. The height of the under-side of the clouds (the cloud base) is found at night by pointing a searchlight vertically upwards and measuring the elevation of the spot of light by means of a sextant. If the observer is 800 yds. from the

searchlight and the angle of elevation is 61°, what is the height of the cloud base ?

2. A man is descending vertically at a constant speed of 20 ft. per second with a parachute. In one minute his angle of elevation changes from 66° to 55°. How much longer will he take to reach the ground ? How far is the observer from the point of landing ?

3. An aircraft flies on a bearing of 071° for 110 miles, then on a bearing of 321° for 90 miles. In what direction must he fly to reach his base direct ?

4. A bomber must release a bomb before the target is reached, because the bomb even while falling continues to travel horizontally with the speed of the bomber. A bomb falls $16t^2$ feet vertically in t seconds, and, if the bomber is flying at 300 m.p.h. the bomb will move $440t$ feet horizontally. If the bomber B is flying at 10,000 ft., find (i) the time the bomb takes to fall, (ii) the angle of depression HBT of the target T from the bomber.

[Air resistance is ignored here.]

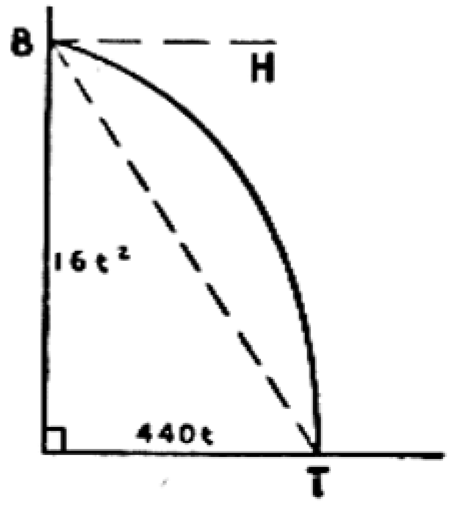

FIG. 80.

CHAPTER V

THE SECANT AND COSECANT

SINCE the sides of a right-angled triangle can be taken two at a time in three ways, there will be six trigonometrical ratios. Four have already been named ; the two remaining ratios are now described.

Secant and Cosecant

For any value of θ, whether acute or obtuse,

$\dfrac{1}{\cos\theta}$ is called the secant of θ or, for short, sec θ ;

$\dfrac{1}{\sin\theta}$ is called the cosecant of θ or, for short, cosec θ.

It will be seen that cosec θ = secant of complement of θ. Cosec is often printed " csc ", so that there is a three-letter abbreviation for each of the six ratios.

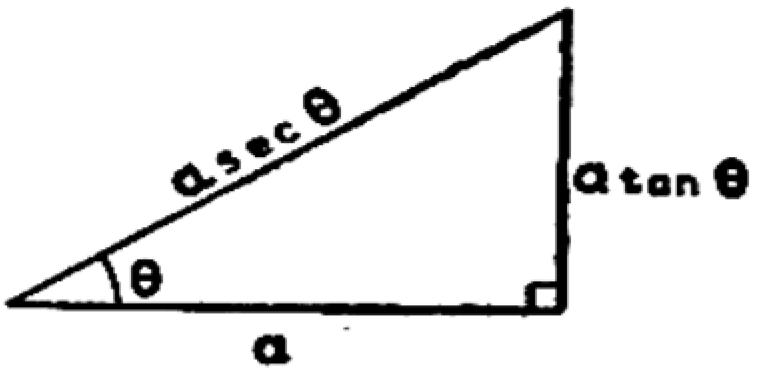

FIG. 81.

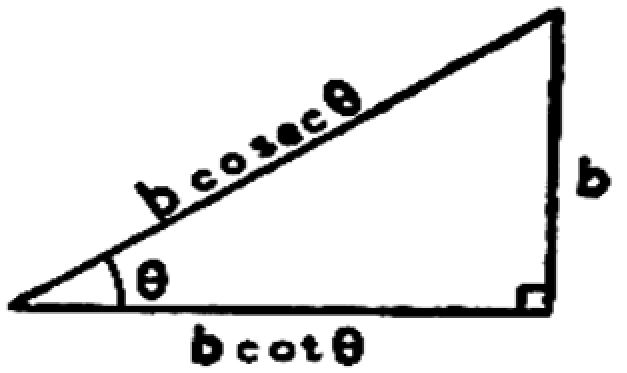

FIG. 82.

For an acute angle, $\sec\theta = \dfrac{\text{hyp.}}{\text{adj.}}$ and $\operatorname{cosec}\theta = \dfrac{\text{hyp.}}{\text{opp.}}$

If the adjacent side is a the hypotenuse is a sec θ (Fig. 81).

If the opposite side is b the hypotenuse is b cosec θ (Fig. 82).

So these ratios are used to find the length of the hypotenuse.

Note that both secant and cosecant must be greater than unity.

Note also that, like the other co-ratios, cosec θ diminishes as θ increases and the numbers in the difference columns of the tables must be subtracted.

Examples—32

Finding sec and cosec

1. If the acute angles of the triangle whose sides are 5 in., 12 in., 13 in. are α and β (where $\alpha < \beta$), write down the secant and cosecant of each of these angles.

2. The sides of a triangle are 12 cm., 12 cm., 8 cm.; what is the secant of a base angle?

3. For obtuse angles, what are the signs of the secant and of the cosecant?

4. BOA is the diameter of a circle of centre O; the perp. bisector of OB meets the circle at X. What is the secant of the obtuse angle AOX?

5. A cone has vertical height 5 in. and slant height 7 in. What is the cosecant of the angle which the slant height makes with the base?

Using sec and cosec

6. A pole is stayed by a wire fastened to the pole at 25 ft. up and fastened to a peg in the ground. What is the length of the wire

(i) If it makes 28° with the vertical? (Use secant.)
(ii) If it makes 56° with the horizontal? (Use cosecant.)

7. The shadow of a vertical stick is 7 ft. long; the elevation of the sun is 33°40′. How far is it from the top of the stick to the end of the shadow?

8. A weight fastened to a string swings round in a horizontal circle which is 5 ft. below the point to which the other end of the string is fixed. What is the length of the string, if it makes 27°18′ with the vertical?

9. The altitude from A to BC of $\triangle ABC$ is 7·6 cm. and it makes 17° and 29°32′ with AB and AC. How long are these sides?

10. Going along a road in direction ENE I reach a point a mile north of my starting point. How far have I gone?

11. The angle at the centre standing on a chord of a circle is 148°; if the length of the chord is 9 in., find the diameter of the circle.

12. In $\triangle ABC$ the perp. from B to CA is 2 in. long and the angles C and A are 43°17′ and 61°32′; find BC and BA.

Pythagoras

13. Applying Pythag. to Fig. 81, p 66, prove that

$$sec^2\ \theta = 1 + tan^2\ \theta.$$

14. Applying Pythag. to Fig. 82, p. 66, prove that

$$cosec^2\ \theta = 1 + cot^2\ \theta.$$

[These relations, with $\cos^2\theta + \sin^2\theta = 1$, are the three trigonometrical versions of Pythagoras' theorem. You should be able to read them quickly from a figure if you do not learn them by heart.]

15. Use the relation $\sec^2\theta = 1 + \tan^2\theta$ to find (i) $\sec\theta$, if $\tan\theta = 1$, (ii) $\tan\theta$, if $\sec\theta = 1{\cdot}25$, (iii) $\tan\theta$, if $\sec\theta = 2$.

16. Use the relation $\text{cosec}^2\theta = 1 + \cot^2\theta$ to find (i) $\text{cosec}\ \theta$, if $\cot\theta = 2$, (ii) $\cot\theta$, if $\text{cosec}\ \theta = 3$, (iii) $\cot\theta$, if $\text{cosec}\ \theta = \frac{17}{15}$.

17. Examine whether the relations of Ex. 13 and 14 are true if θ is obtuse.

18. Show that $\sec^2\theta - \text{cosec}^2\theta = \tan^2\theta - \cot^2\theta$, and verify by finding numerical values when $\sin\theta = \frac{4}{5}$ and θ is obtuse.

19. (i) By what must each term of $\sec^2\theta = 1 + \tan^2\theta$ be multiplied to give the result $1 = \cos^2\theta + \sin^2\theta$?

(ii) By what must each term of $\cos^2\theta + \sin^2\theta = 1$ be divided to give the result $\text{cosec}^2\theta = 1 + \cot^2\theta$?

20. Turn the relation $\sec^2\theta = 1 + \tan^2\theta$ into a relation between $\text{cosec}\ \phi$ and $\cot\phi$, where $\theta + \phi = 90°$.

Miscellaneous

21. (i) If $\sec\theta - \text{cosec}\ 21° = 0$, find θ.

(ii) If $\text{cosec}\ \phi + \sec 62°15' = 0$, find ϕ.

22. Look out sin 42° and, without further reference to the tables, find sec 48°.

23. The height of an equilateral triangle is 10 cm. Find the side of the triangle and the diameter of its circum-circle.

24. Taking 1 in. as unit for $\text{cosec}\ \theta$, draw a graph for values of $\text{cosec}\ \theta$ less than 5, between 0° and 180°, using the tables.

25. Draw a graph of $\sec\theta$ from $\theta = 0$ till $\sec\theta = 4$, taking 1 in. as unit. Sketch on a smaller scale how the graph from 90° to 180° is related to that from 0° to 90°.

Practice in Using the Six Ratios

When all six ratios are in use, much practice is needed if the danger of frequent confusion between one and another is to be avoided. This may be obtained (i) by naming the sides which give each of the ratios for right-angled triangles, placed and lettered in various ways, (ii) by finding expressions for the sides of various figures, one side being given, (iii) by simplifying expressions containing the ratios by means of the relations between them.

Naming Ratios **Examples—33**

1. In Fig. 83, ABCD and PQRS are rectangles.

(i) What are sin θ, tan θ, sec θ? *E.g.*, $\cos\theta = \frac{AQ}{AB}$.

(ii) Which other angles are equal to θ? For each name the secant.

(iii) What are cot QPR and cosec PRQ?

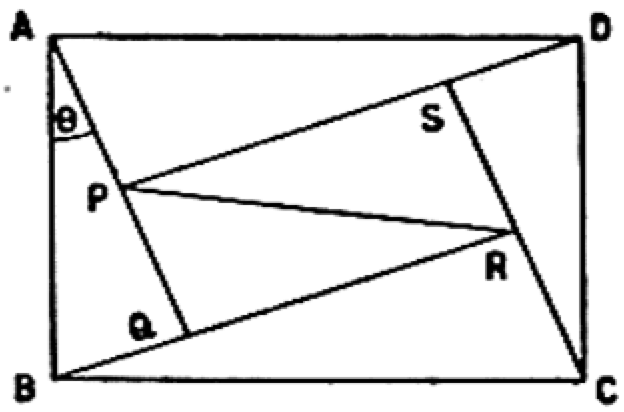

FIG. 83.

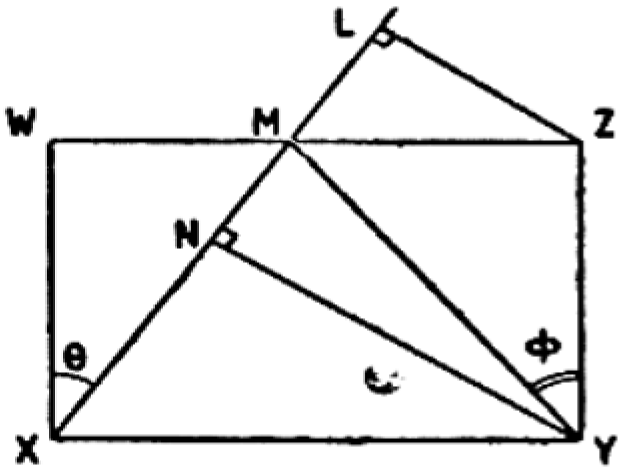

FIG. 84.

2. In Fig. 84, WXYZ is a rectangle. (i) What are sec θ, cot θ, cosec ϕ, cos ϕ? (ii) Which other angles are equal to θ? Name the sine of each. (iii) The figure shows three angles each $90° - \theta$; name them and their secants.

3. Two circles, centres A and B, meet at C, D. CD meets AB at N between A and B. $\text{Cos CAN} = \frac{AN}{AC}$; state similarly cosec CAN, tan ACN, sin BCN, cot CBN. If $\angle ACB$ is a right angle give second answers for two of these.

4. TA, TB are tangents to a circle centre O; AB meets OT at S; $\tan ATO = \frac{AS}{ST} = \frac{OA}{AT}$. In a similar way give two values for sec AOT and cot AOT.

Expressions for Sides

5. ABC is a triangle right-angled at B. P is the foot of the perpendicular from B to AC; M is the foot of the perpendicular from P to AB. If MA $= c$ and $\angle$BAC $= \alpha$, find expressions for PA, BA, CA, BM, CP in terms of c and $\sec \alpha$ and $\tan \alpha$.

6. In Fig. 83, AB $= a$ and AD $= b$; find expressions in turn for the lengths AQ, BQ, AP, PD, PQ, PS. Show in two ways that

$$\text{area PQRS} = ab - (a^2 + b^2) \cos \theta \sin \theta.$$

7. In Fig. 84, a is the side of the square WXYZ. Find expressions for all the lengths in the figure and show that $\tan \phi = 1 - \tan \theta$ and that

$$\tan (\theta + \phi) = \cos \theta / (\sec \theta - \sin \theta). \quad [\text{Use } \triangle\text{NMY.}]$$

8. ABC is an equilateral triangle and $\angle$BAC is divided into four equal parts by the lines AE, AF, AG; B, E, F, G, C being in this order on BC. Name pairs of lines whose ratios are sec 15°, cosec 15°, cot 30° and draw a line BM whose length is AB sin 45°. Why is AB sin 45° equal to BG sin 75°?

9. ABC is a triangle with a right-angle at A, and D is the foot of the perpendicular from A to BC. If $\angle$ABD $= \theta = \angle$DAC, express AB, BD, AC, CD in terms of a or BC and θ, and use the results to show that

$$\cos^2 \theta + \sin^2 \theta = 1.$$

10. Using the figure of No. 9, express the four lengths in terms of h and θ, where $h =$ AD. Hence show that

$$(\cot \theta + \tan \theta)^2 = \text{cosec}^2\, \theta + \sec^2 \theta.$$

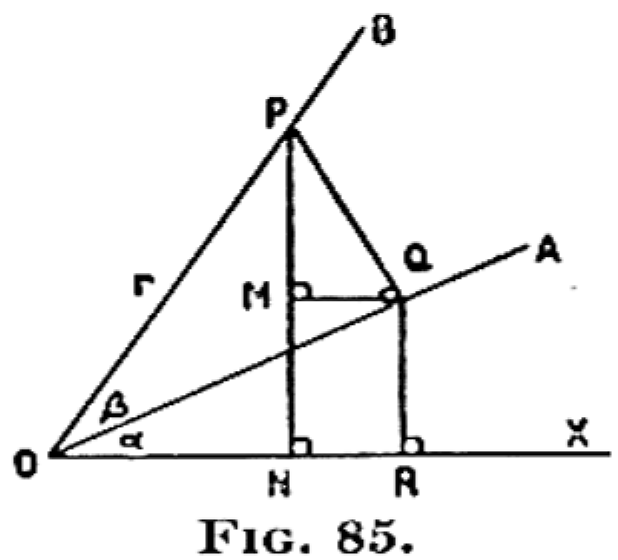

FIG. 85.

11. In Fig. 85, $\angle$AOX $= \alpha$, $\angle$BOA $= \beta$; PN, PQ, QM, QR are perpendicular to OX, OA, PN, OX respectively. If OP $= r$ prove that $<$MPQ $= \alpha$ and express the lengths of PQ, OQ, OR, QR, PN, ON, PM, NR in terms of r, α and β.

12. Use the relation PN $=$ PM $+$ QR, with the data and results of Ex. 11, to prove that

$$\sin (\alpha + \beta) = \sin \alpha \,.\, \cos \beta + \cos \alpha \,.\, \sin \beta.$$

13. Use the relation ON = OR — MQ, with the data and results of Ex. 11, to prove that

$$\cos(\alpha + \beta) = \cos\alpha \,.\, \cos\beta - \sin\alpha \,.\, \sin\beta.$$

14. If, in Fig. 85, PN $= h$, express the lengths of OP, PQ, PM in terms of h, α, β.

Relations between the Six Ratios

The six trigonometrical ratios of a given angle are connected with each other in such a way that, if any one ratio is known, the others can be found. The relations can be divided into two groups :—

I. Definition Relations.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Each of these can be used as a definition in the case of obtuse angles.

II. Pythagorean Relations.

$\cos^2\theta + \sin^2\theta = 1 \quad \sec^2\theta = 1 + \tan^2\theta \quad \operatorname{cosec}^2\theta = 1 + \cot^2\theta.$

Each of these is a version of Pythagoras' theorem.

Other results can be deduced from the above relations, and complicated expressions can often be simplified or made easier for numerical calculation. If in doubt when attempting a simplification, express the other ratios in terms of sine and/or cosine.

Examples—34

Simplify the expressions in Ex. 1 to 4.

1. (i) $\tan\theta\cot\theta$. (ii) $\sin\theta\sec\theta$. (iii) $\sin^2\theta\operatorname{cosec}\theta$. (iv) $1/\sec\theta$.

2. (i) Sec A cot A. (ii) $\sec^2 A\cos A$. (iii) $\tan A\sin A\operatorname{cosec}^2 A$. (iv) $\sec A/\operatorname{cosec} A$.

3. (i) $1 - \cos^2\theta$. (ii) $1 - \sin^2\theta$. (iii) $\cos^4\theta + \cos^2\theta\sin^2\theta$. (iv) $\sec^3\theta - \tan^2\theta\sec\theta$.

4. (i) $\dfrac{\cos^2 A}{\sin^2 A} + 1$. (ii) $\dfrac{1}{\sin^2 A} - 1$.

(iii) $\dfrac{\sin^2 A}{\cos^2 A} + 1$. (iv) $\dfrac{1}{\tan^2 A} + 1$.

5. Use the diagrams in Fig. 86 to show that $\sin 30° = \frac{1}{2}$ and that $\tan 45° = 1$. Then calculate the values of the six ratios for each of the angles 30°, 45°, 60° and show the results in a table. Do not work out any square roots.

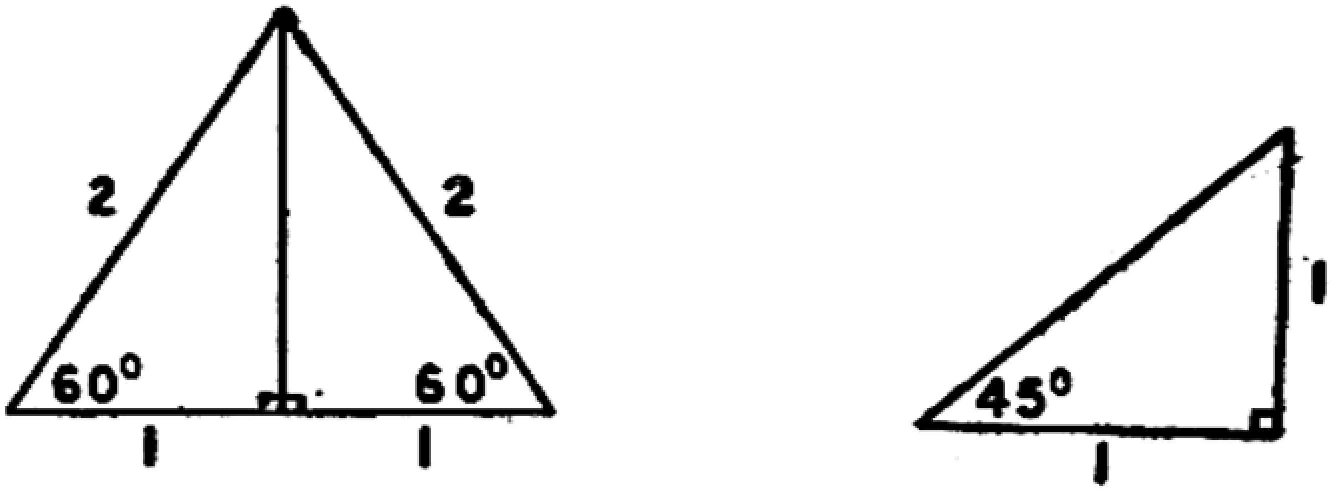

FIG. 86.

6. Prove that $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$.
7. Prove that $(1 + \tan B)^2 = \sec^2 B + 2 \tan B$.
8. Prove that $\sin^2 B - \cos^2 A = \sin^2 A - \cos^2 B$.
9. If $x = p \cos \theta$ and $y = p \sin \theta$ show that $x^2 + y^2 = p^2$.
10. If $x = q \sec \theta$ and $y = 2q \tan \theta$ show that $x^2 - \frac{1}{4}y^2 = q^2$.
11. Find $\tan \theta$ if $\sin \theta = 4 \cos \theta$, and hence find θ.
12. Find $\tan \theta$ if $9 \sin^2 \theta - \cos^2 \theta = 7$.
[Write $7 \cos^2 \theta + 7 \sin^2 \theta$ for 7, and divide both sides by $\cos^2 \theta$.]
13. Prove that $(\cot \theta + \tan \theta)^2 = \operatorname{cosec}^2 \theta + \sec^2 \theta$, an identity shown geometrically in Examples 33, No. 10. (p. 70).
14. If $x = \cos \theta \cos \phi$, $y = \sin \theta \cos \phi$, $z = \sin \phi$, prove $x^2 + y^2 + z^2 = 1$.
15. If $m = \sec \theta + p \tan \theta$, $n = \tan \theta + p \sec \theta$, prove $m^2 - n^2 = 1 - p^2$.

Inverse Notation

It is found convenient to have an abbreviation for the phrase "the angle whose sine is x."

The abbreviation used is $\sin^{-1} x$. The advantage is that the sentence "if $\sin \theta = {\cdot}4321$, then $\theta = 25°36'$" can be replaced by "$\sin^{-1} {\cdot}4321 = 25°36'$."

The notation, though a little puzzling at first, is strictly logical.* The idea is that "$\sin^{-1}$" undoes the effect of

---

* It is the abbreviation $\sin^2 \theta$ for $(\sin \theta)^2$ that is not really logical. And $\sin^{-1} \theta$ is not needed for $1/\sin \theta$ as we use $\operatorname{cosec} \theta$ for this.

"sin," and *vice versa*, and a little reflection should convince you that

$$\sin^{-1}(\sin 42^\circ) = 42^\circ, \text{ and } \sin(\sin^{-1} \cdot 4826) = \cdot 4826.$$

In the same way $\cos^{-1}$, $\tan^{-1}$, etc., are used, for example :—

$$\cos^{-1} \cdot 5 = 60^\circ,\ \tan^{-1} 1 = 45^\circ,\ \sec^{-1} 2{\cdot}4022 = 65^\circ 24'.$$

Care is needed, however, in using the $\sin^{-1} x$ notation if the angles are not necessarily acute. There are *two* angles, one acute and one obtuse, which have a given sine. For example, $\sin^{-1} \frac{1}{2}$ might mean either 30° or 150°, and the same is true of $\operatorname{cosec}^{-1} 2$.

When you have reached the definitions of the ratios for angles which may be greater than 180°, you will find that *all* the inverse functions are many-valued. But *if you are not told otherwise* you should take $\sin^{-1} x$ to mean the *smallest* angle whose sine is x.

Examples—35

1. Sketch $\sin^{-1} \frac{1}{3}$, $\cos^{-1} \frac{3}{4}$, $\tan^{-1} 2$, and, using squared paper, draw and measure $\cos^{-1} \cdot 7$ and $\sin^{-1} \cdot 36$.

2. Write down the values of $\tan(\tan^{-1} \frac{1}{2})$ and $\cos^{-1}(\cos 20^\circ)$.

3. Find from the tables (i) $\sin^{-1} \cdot 9063$. (ii) $\cos^{-1} \cdot 9690$. (iii) $\tan^{-1} 1{\cdot}3319$. (iv) $\tan^{-1} \cdot 5965$. (v) $\cos^{-1} \cdot 2851$. (vi) $\sin^{-1} \cdot 5100$.

4. Verify that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} 1$, and that $\sec^{-1} 2{\cdot}2812 + \cot^{-1} 2{\cdot}0503 \simeq 90^\circ$, as closely as the tables can show.

5. What is the difference between the two values of $\sin^{-1} \cdot 7660$ which are less than 180° ?

6. Show that (i) $\cos^{-1} \frac{1}{2} + \cos^{-1}(-\frac{1}{2}) = 180^\circ$. (ii) $\tan^{-1} 2 = 90^\circ - \cot^{-1} 2$. (iii) $\tan^{-1} 2 = \cot^{-1} \frac{1}{2}$.

7. What are the values of (i) $\tan^{-1} 1\frac{1}{3} + \tan^{-1} \frac{3}{4}$. (ii) $\sin^{-1} 1 - \sin^{-1} \frac{1}{2}$. (iii) $\sin(\operatorname{cosec}^{-1} 3)$. (iv) $\cos^{-1}(\sin 30^\circ)$. (v) $\tan(\cot^{-1} 1)$?

Contours

If contours are marked on a map, giving differences in height of h ft., and if two adjacent contours are b ft. apart, then the average slope between is θ, where $\cot\theta = \frac{b}{h}$ and the distance between (if the slope is uniform) is $b \sec \theta$, which is equal to $h \operatorname{cosec} \theta$. Thus, in Fig. 87, the average slope from A to B is given by $\cot \theta = \dfrac{220}{100} = 2{\cdot}2$; so $\theta \simeq 24\frac{1}{2}°$ and the distance AB is $100 \operatorname{cosec} 24\frac{1}{2}° \simeq 240$ ft.

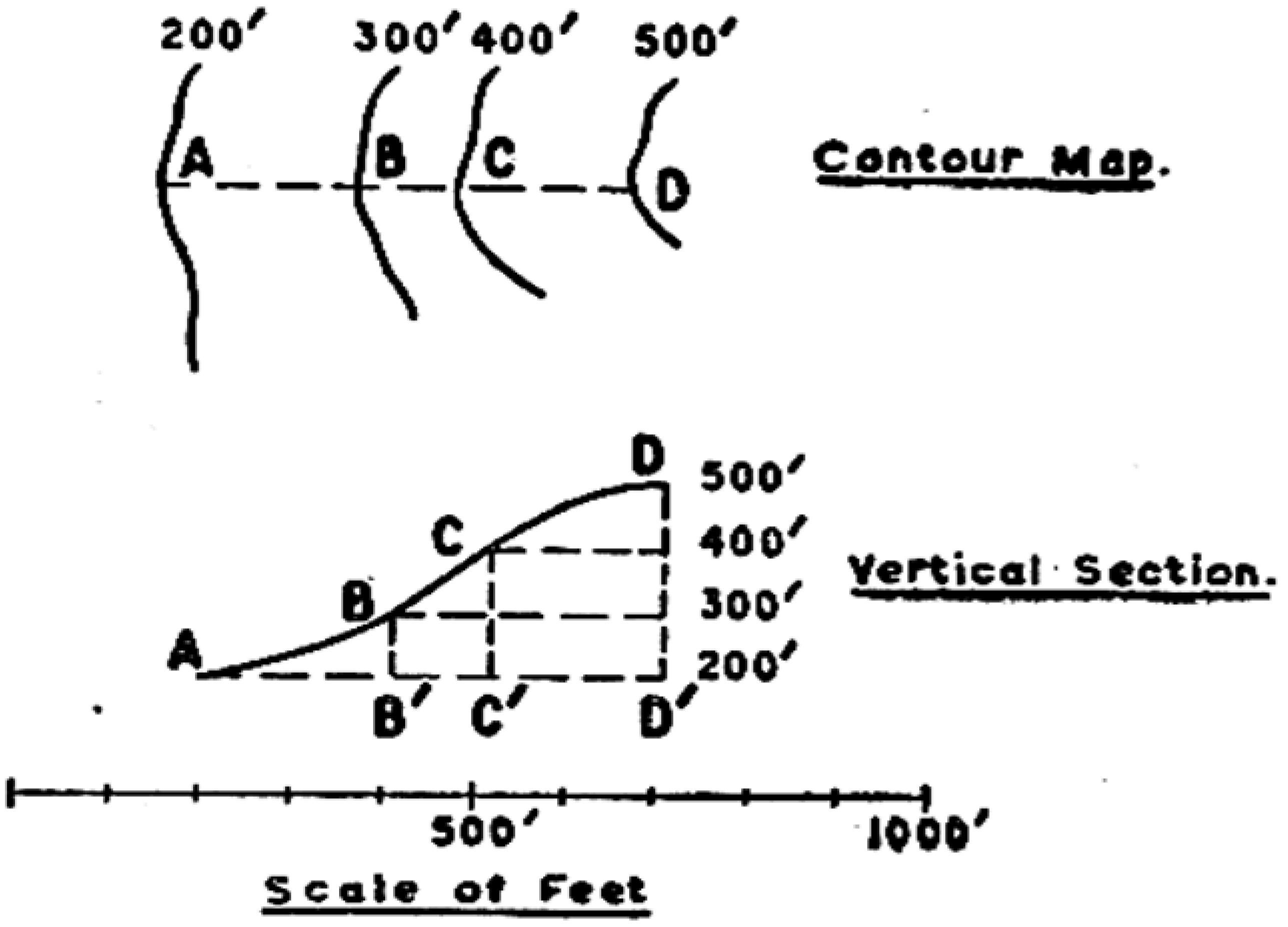

Fig. 87.

Contours are usually drawn at 100 ft. apart and the horizontal scale is not given in feet but in miles. You must remember that the eye exaggerates vertical distances, so that a slope of 20° is very steep, and a hill as steep as an ordinary staircase appears to be almost a precipice.

Examples—36

1. In Fig. 87, find the average slopes from B to C and from C to D and the distances along the slopes, supposed uniform.

2. A staircase has 9 in. depth of treads, 6 in. height of steps. Find its angle of slope. A hill is of this steepness; how high would it rise in 100 yds. along the slope? and what would be the horizontal distance covered?

3. In a map on the scale 6 in. to the mile the contours appear as in Fig. 87. If the measurements are AB = 0·5 in., BC = 0·2 in., CD = 0·3 in., and if in each case the slopes are taken as uniform, show that (i) the steepest slope is about $29\frac{1}{2}°$, (ii) from A to B the distance along the slope is only about 10 ft. more than the horizontal distance.

The Plane Hillside

Inferences from contour maps are only approximate; there may be a 50-ft. precipice between the 300-ft. and 400-ft. contours in Fig. 87. If, however, a hillside can be treated as a plane, important relations can be found between various angles concerned. These apply to a variety of cases, such as sloping roofs, sloping desks, shadows of walls, etc.

THE HILLSIDE PROBLEM

The figure of the plane hillside (Figs. 88A and 88B) needs to be looked at carefully. ABEF is the face of the slope, ABCD is horizontal underneath it. AF is a line of greatest slope, ∠DAF the angle of greatest slope. AE is a path going up the hill at a slant and its angle of slope β is of course less than α.

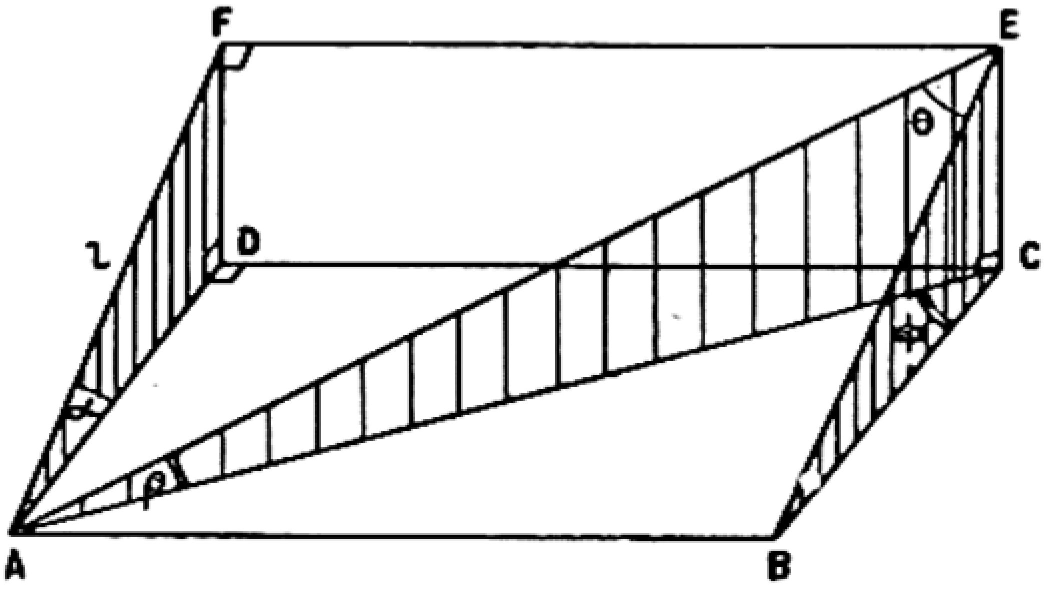

FIG. 88A.

The various lines from B are drawn to make the figure clearer; $\angle EBC = \alpha$. You may find it easier to see the various angles in Fig. 88B.

The first point to notice is that ∠s FAE and DAC are not the same. To see this, note that ∠s EFA, CDA are right angles, so that

$$\tan DAC = \frac{DC}{DA}, \qquad \tan FAE = \frac{FE}{FA}.$$

You will see that $DC = FE$, but $DA < FA$,

$$\therefore \tan FAE < \tan DAC.$$

Call these angles θ, ϕ. It is convenient to mark them on the alternate angles at E and C.

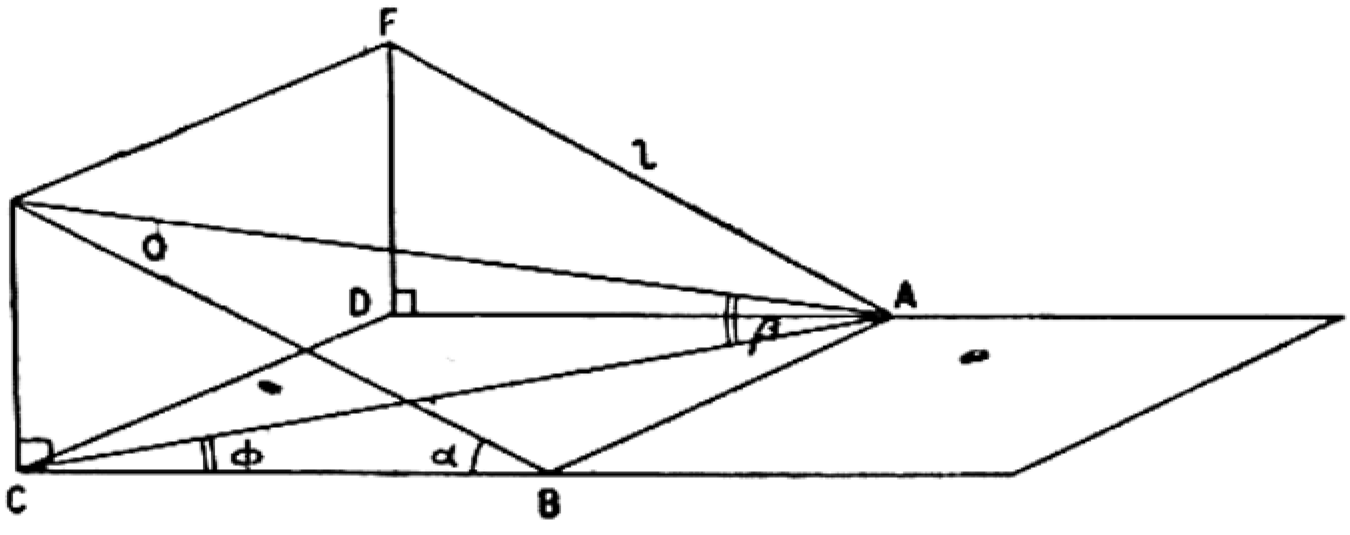

FIG. 88B.

Note that ∠FAE is the angle the slanting path AE makes with the line of greatest slope AF; while ∠CAD, which is the angle between the planes EAC and FAD, is the difference of the compass bearings of E and F as seen from A. You have seen that this is greater than ∠FAE.

Starting with the length of AF (taken as l), all other lengths can be put down in terms of l and the angles. *E.g.*, AD $= l\cos\alpha$ from △FAD.

Examples—37 (see Fig. 88A)

1. Write down the length of FD in terms of l and α, and of FE and AE in terms of l and θ.

2. If AD $= d$, write down the lengths of DC and AC in terms of d and ϕ. Now use $d = l\cos\alpha$ to express them in terms of l, α, ϕ.

3. On a hillside sloping at 20°, a slanting path makes 48° with the line of greatest slope, which runs due north. Find the slope of the path and its compass bearing.

[Sol'n. In Fig. 88A $\alpha = 20°$, $\theta = 48°$. If AF $= l$, AD $= l\cos 20°$, FD $= l\sin 20°$, from △FAD.

$\bar{1}{\cdot}5341$ From △FAE, FE $= l\tan 48°$, AE $= l\sec 48°$.

$\bar{1}{\cdot}8255$ ∴ in △EAC, EC $=$ FD $= l\sin 20°$, AE $= l\sec 48°$.

$\bar{1}{\cdot}3596$ ∴ $\sin\beta = \dfrac{l\sin 20°}{l\sec 48°} = \sin 20°\cos 48°$.

$$\therefore\ \beta = 13°\ 14'.$$

Again DC $=$ FE $= l\tan 48°$, DA $= l\cos 20°$.

$0{\cdot}0456$ Hence in △DAC,

$\bar{1}{\cdot}9730$

$0{\cdot}0726$

$$\tan \text{DAC} = \frac{\text{DC}}{\text{DA}} = \frac{l\tan 48°}{l\cos 20°} = \frac{\tan 48°}{\cos 20°}.$$

$$\therefore\ \phi = \angle\,\text{DAC} = 49°46'.$$

The slope of the path is 13°14′ and its bearing is 49°46′.

4. Repeat Ex. 3, taking the slope of the hillside to be 26°30′ and the path to make 56° with the line of greatest slope.

5. In Fig. 88A, FDCE is a wall running east and west. The shadow of the point E falls at A; AF runs due south; AC runs 50° west of south. The wall is 15 ft. high and A is 27 ft. from the wall. Find the angle of elevation of the sun, the angle DAF and the angle between FA and EA.

[Hint. First find α from $\tan\alpha = \frac{15}{27}$, $DC = 27 \tan 50°$ and $AC = 27 \sec 50°$.

The answers are $\alpha = 29°3'$, $DC = 32{\cdot}18$, $AC = 42{\cdot}01$.

Then β is found from $\triangle EAC$ $\left(\tan\beta = \frac{EC}{AC}\right)$.

Also AF is found from $\triangle FAD$ and $\tan FAE = \frac{FE}{AF}$.

The solution should be written out in full.]

6. A wall 12 ft. high runs east and west. The sun is 43° west of south at an elevation of 46°. Find the width of the wall's shadow.

[Hint. Here EC, β, ϕ, are given, and AD is wanted. Find AC from $\triangle EAC$ and then AD from $\triangle DAC$.]

7. The sun is 36° east of south and an east-and-west wall 10 ft. high has a shadow whose width is 7 ft. Find the elevation of the sun.

[Hint. East or west makes no difference to the answer here, as the elevation of BF is the same as that of AE; ϕ is given and $\cot\alpha = 0{\cdot}7$. Prove that $\tan\beta = \frac{10}{7}\cos\phi$.]

8. On a plane hillside sloping at 24° a slanting path slopes at 17°. What angle does this path make with the line of greatest slope?

[Hint. In Fig. 88A, $\alpha = 24°$, $\beta = 17°$, θ is wanted. If $AF = l$, $EC = FD = l \sin 24°$. Hence $AE = l \sin 24° \operatorname{cosec} 17°$. But $\frac{AE}{AF} = \sec\theta$.]

General Results

If you look back at Nos. 3 to 8 in the above Examples, you will see that they are all about the four angles α, β, θ, ϕ of Fig. 88.

In Nos. 3 and 4, α and θ are given; β and ϕ are to be found.

In Nos. 5 and 7, α is easily found, ϕ is given; and (in 5) θ and β are to be found.

In No. 6, β, ϕ are given; α is to be found.

It is often easier in such cases to find the general formula first and afterwards to put in the special numerical values.

Examples—38

1. Given ϕ, find formulæ for $\tan\beta$ and $\tan\theta$.

[Hint. As ϕ is given, start with AD $= a$ and put down the lengths of EC (equal to DF) and AC for β and of FE (equal to DC) and AF for θ].

On a plane hillside of slope 25°, the line of greatest slope being on a bearing of 20°, I walk on a bearing of 64°. At what slope do I rise?

2. Given α, θ prove that $\sin\beta = \sin\alpha\cos\theta$ and that $\tan\phi = \tan\theta\sec\alpha$.

A roof whose inclination to the horizontal is 40° meets another part of the roof on a line which makes an angle of 28° with the line of the greatest slope of the first roof. At what angle is this line inclined to the horizontal?

3. Given β, ϕ, prove that $\cot\alpha = \cot\beta\cos\phi$. [Start with EC $= h$.]

A wall 12 ft. high runs east and west; the sun is at an altitude of 32° and in a direction 47° west of south. How wide is the shadow of the wall?

4. Given α, β find formulæ for $\cos\theta$ and $\cos\phi$.

[Hint. If AF $= l$, EC $= l\sin\alpha$ and EA $= l\sec\theta$, hence $\sin\beta = \sin\alpha/\sec\theta$.]

On a plane sloping at 42°, a line is drawn sloping at 21°. What angle does this make with the line of greatest slope and what is the difference of their compass bearings?

The Circle. Arcs, Sectors, Segments

If a circle has diameter d, the length of the circumference C is found from the formula $C = \pi d$. In terms of the radius, $C = 2\pi r$. The value of π has been calculated to 707 decimal places, but for practical purposes the following approximations are used :—

$3\frac{1}{7}$ (too large), 3·14 (too small), 3·1416 (slightly too large).

To 50 places, π = 3·14159 26535 89793 23846 26433
83279 50288 41971 69399 37510.

$$\text{Log}\,\pi \simeq 0{\cdot}49715, \qquad \text{to 4 fig.} = 0{\cdot}4971.$$

From the symmetry of the circle it is clear that the length of any arc is proportional to the angle subtended at the centre.

Arc : circumference = angle in degrees : 360°.

$$\therefore \text{ length of arc subtending } \theta^\circ \text{ at the centre} = \frac{\theta C}{360}.$$

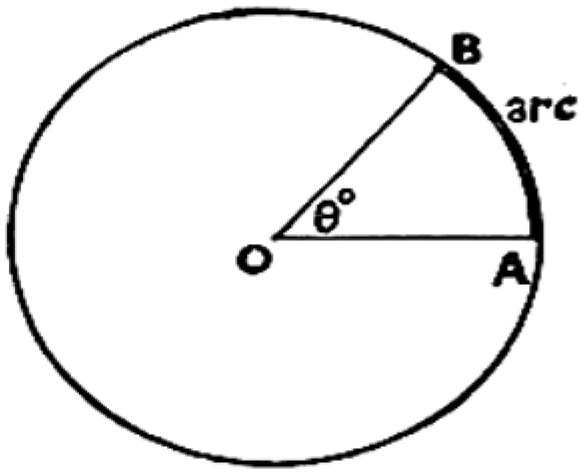

FIG. 89A.

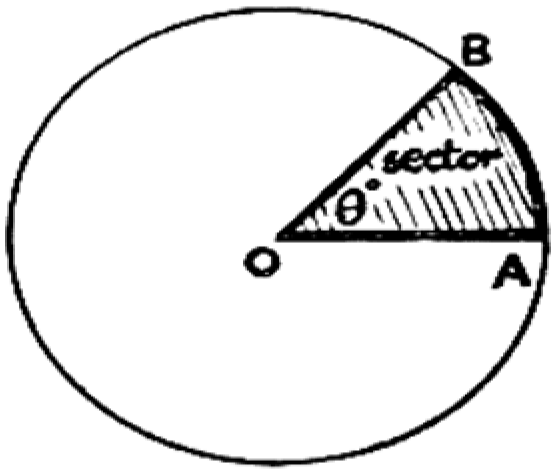

FIG. 89B.

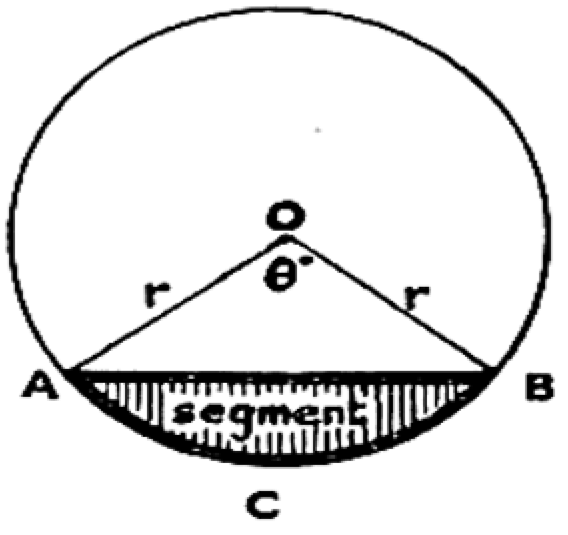

FIG. 90.

It can be proved that the area of a circle is equal to the radius multiplied by half the circumference, or $S = \frac{1}{2}Cr$.

$$\therefore \text{ The area of the circle } S = \pi r^2 \text{ or } \tfrac{1}{4}\pi d^2.$$

The area of a sector of a circle (OAB in Fig. 89B) is proportional to the angle AOB at the centre. If S is the area of the whole circle, the area of a sector whose angle is θ° is $\frac{\theta S}{360}$, or radius × half the arc.

The area of a segment of a circle (ABC in Fig. 90) is found by subtracting the area of the triangle OAB from that of the sector OACB.

Examples—39

Use either $\pi = 3{\cdot}1416$ or $\log \pi = 0{\cdot}4971\frac{1}{2}$. ($\log 2\pi \simeq 0{\cdot}7982$.)

1. (i) Find the length of the arc of a circle of radius 10″ which subtends 60° at the centre. (ii) Find the angle at the centre if the arc is of length 10″.

2. If, in Fig. 89B, the radius is 10″ and $\theta = 60°$, what is the difference in area between the sector OAB and the triangle OAB ?

3. Taking the radius of the earth as 3960 miles, find the distance (along the earth) from the equator to London. Fig. 89A will serve; if B represents London, A is on the equator, and $\theta = 51\frac{1}{2}°$.

4. Taking Fig. 90 as the section of a railway tunnel, find the shaded area if $AB = 8$ ft. and the height of the top of the tunnel above AB is 8 ft. [First show that the radius is 5 ft.; then find θ.]

5. A taut string passes round a wheel of radius 5″ and round a pin distant 10″ from the centre of the wheel. Find the length of the string. [The string is in three parts, a major arc and two tangents.]

The Sphere

Any section of a sphere by a plane is a circle. If the plane passes through the centre of the sphere, the circle is called a "*great circle*": if the plane does not pass through the centre of the sphere, the section is called a "*small circle*." Fig. 91 shows a sphere cut by two parallel planes, giving a great circle with centre O and a small circle with centre C.

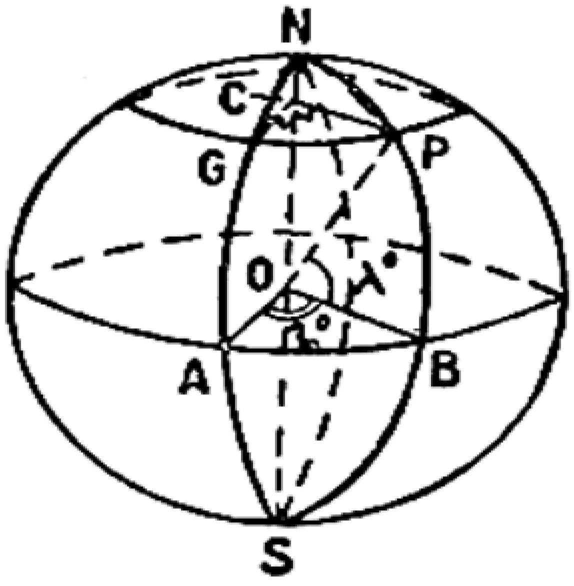

FIG. 91.

The Earth is almost exactly a sphere. If Fig. 91 represents the Earth, and the circle, centre O, is the Equator, then the circle, centre C, is a *parallel of latitude*.

A semicircle whose ends are at the North and South poles is called a *meridian*, since all points on it have midday at the same time. If P is a point on the small circle, centre C, and the meridian NPS cuts the equator at B, the angle BOP is the *latitude* of P. All points on the same parallel have the same latitude.

The meridian through Greenwich (London) is taken as the standard. If G represents Greenwich in Figs. 91 and 92, then the angle GCP (or the equal angle AOB) is called the *longitude* of P. Fig. 92 gives a view in the direction of the Earth's axis.

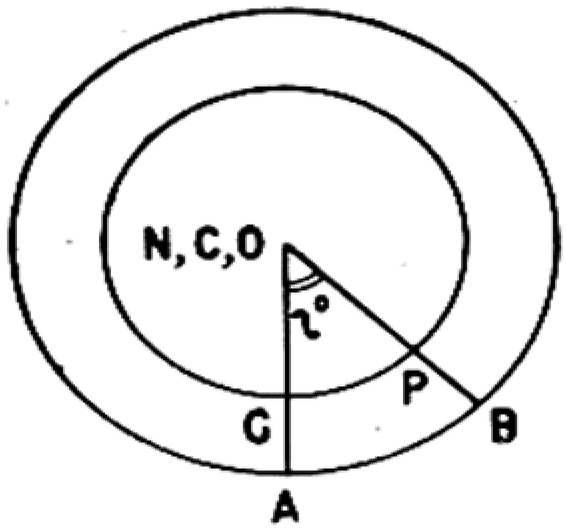

FIG. 92.

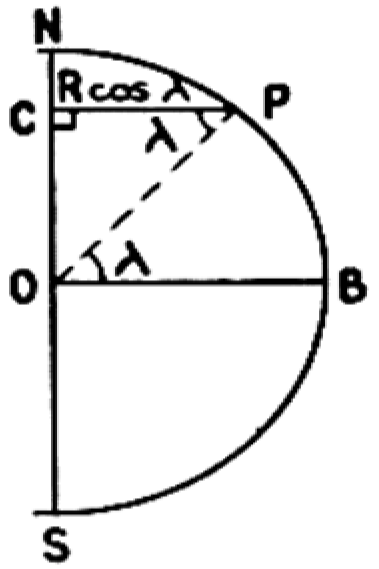

FIG. 93.

The position of any point on the Earth's surface is fixed by its latitude ($\lambda°$ N. or S. of the equator) and its longitude ($l°$ E. or W. of Greenwich).

If the Earth's radius is R miles, the radius of the parallel of latitude λ is $CP = R\cos\lambda$ (see Fig. 93), and its circumference is $2\pi R\cos\lambda$, which is $C\cos\lambda$ if C is the circumference of the equator (or any other great circle).

The distance BP (Fig. 93) along the meridian is $\frac{\lambda C}{360}$ miles.

,, ,, GP (Fig. 92) ,, ,, parallel ,, $\frac{lC}{360}\cos\lambda$,,

For convenience navigators take $C = 21{,}600$ *nautical* miles, since there are 21,600 *minutes* in 360°. With this unit the distances BP and GP become simplified to 60λ and $60l\cos\lambda$ *nautical* miles, and since 1 minute ($1'$) is $\frac{1}{60}$ degree, we get the following simple rules :—

$1'$ change of latitude corresponds to a movement N. or S. of 1 nautical mile.

$1'$ change of longitude corresponds to a movement E. or W. of $\cos\lambda$ nautical mile,
this distance being measured along the parallel of latitude.

[$\cos\lambda$ varies of course from 1 at the equator to 0 at either pole.]

A *nautical mile* is 6080 ft. or 1·1515 ordinary (statute) miles.

A *knot* is a speed of 1 nautical mile per hour.

The lengths of the kilometre, statute mile and nautical mile are as 41 : 66 : 76.

Examples—40

1. Find the distance from Bournemouth to Berwick-on-Tweed, which are almost on the same meridian, and whose latitudes are 50°43′ N, 55°47′ N.

2. Find the distance following the parallel of latitude from Edinburgh to Moscow, whose latitude may be taken as 55·9° and whose longitudes are 3°11′ W. and 37°34′ E.

[Sol'n. The difference of longitude is 40°45′ (note that Edinburgh is *west* of Greenwich) or 2445′. The distance is 2445 cos 55·9 n. miles.]

3. New York and Naples have longitudes 74°00′ W. and 14°15′ E. respectively, and their common latitude is approximately 41° N. What is their distance apart in statute miles following the parallel of latitude.

4. Find the distance from Cape Town (lat. 33°56′ S.) to Danzig (lat. 54°22′ N.) taking them to be on the same meridian (practically $18\frac{1}{2}$° E.).

5. A ship sails from Tynemouth (lat. 55°01′N., long. 1°26′ W.) to Bergen (lat. 60°24′ N., long. 5°18′ E.). Calculate the distance sailed from T to B by using "Pythagoras" on the triangle TXB, where X is the point whose latitude is 60°24′ N. and whose longitude is 1°26′ W. Use nautical miles. Also calculate the distance by using the triangle TYB, where Y is the point whose latitude is 55°01′ N. and longitude 5°18′ E. (Fig. 94).

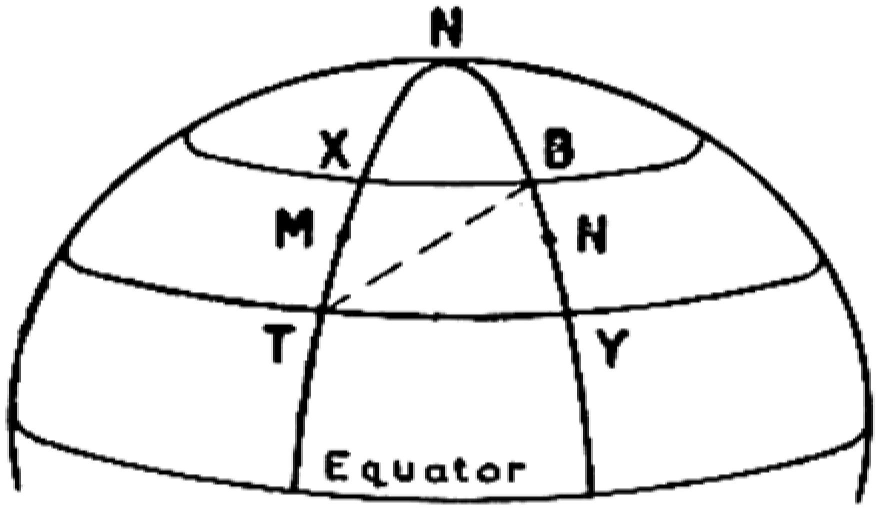

FIG. 94.

6. In order to make a voyage such as that in Ex. 5, a sailor wants to know (i) the course to steer, (ii) the distance. For distances under 500 miles it is found to be sufficiently accurate to use the following rules :—

(i) For the course, tan XTB = MN/TX where MN is on the parallel midway between XB and TY.

(ii) For the distance, TB = TX sec XTB. (MN is called the " mid-latitude departure ").

Calculate the course and distance. [It is assumed that the ship is sailing from corner to corner of a plane rectangle of length MN and width TX, and the method is consequently known as " plane sailing."]

7. Calculate the course and distance for a voyage from T to a point half-way between X and B and on the same parallel, using the method of Ex. 6.

8. In Fig. 91 take $R = 4000$ mi., $l = 90°$, $\lambda = 34°$ and find
(i) The distance GP along the parallel of latitude.
(ii) The straight distance through the Earth from G to P (use Fig. 92).
(iii) the angle GOP.
(iv) The distance from G to P along the great circle.
[Hint. The arc GP subtending angle GOP at O is required.]

9. An aircraft has to fly from A to P in Fig. 91, where $\lambda = 60°$, $l = 60°$. How much further would it be to fly from A to B and thence to P than from A to G and on to P along the parallel of latitude ? Work in nautical miles.

10. With the data of Ex. 9, calculate the distance from G to P, in nautical miles, along the great circle through G and P. [Find $\angle$GOP.]

11. With the data of Ex. 9, draw $PQ \perp OB$ meeting OB at Q and calculate (i) AQ (using Fig. 92), (ii) PQ (using Fig. 93), (iii) AP, from $\triangle$AQP, (iv) Angle AOP, (v) the distance from A to P along a great circle.

N.B. The shortest distance between two points on a sphere is along the great circle through the points (see pp. 80–81).

More Areas and Volumes

The volume of a *prism* or *cylinder* is given by $V = Bh$, where B is the area of the base and h the vertical height.

The volume of a *pyramid* or *cone* is given by $V = \frac{1}{3}Bh$.

The area of the *curved surface of a cone* is got by multiplying the circumference of the base by half the slant height.

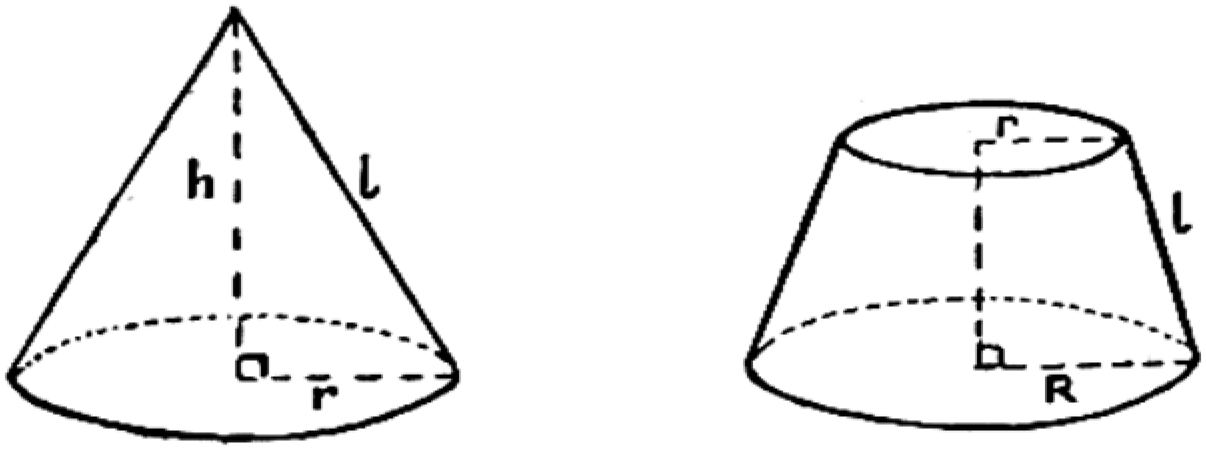

FIG. 95.

$S = \pi rl$ where r is the radius of the base and l the slant height. The area of the curved surface of a *frustum* of a cone is given by $S = \pi(r + R)l$, where r, R are the radii of top and base, and l the slant height (Fig. 95).

Archimedes proved that the area of a *spherical cap* (**Fig. 96**) is equal to that of a cylinder whose height is the vertical height of the cap and whose radius is that of the whole sphere.

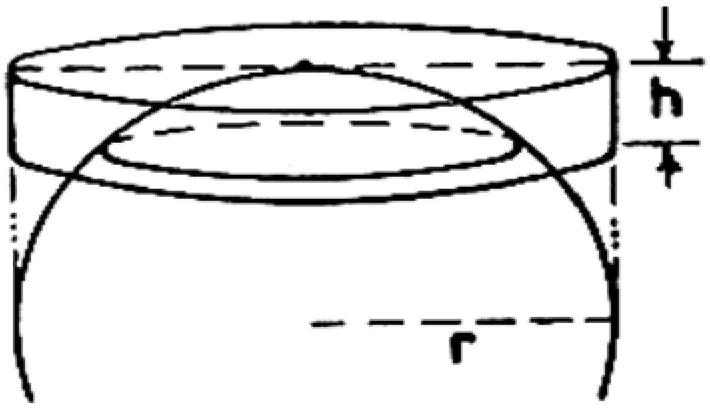

FIG. 96.

$$\therefore \text{ Area of cap} = 2\pi rh.$$

The area of a *zone* of a sphere is the difference of the areas of two caps, so if the distance between its circular ends is h,

$$\text{Area of zone} = 2\pi rh.$$

The area of the whole *sphere* is found by making $h = 2r$.

$$\therefore \text{ Area of sphere} = 4\pi r^2 = \pi d^2.$$

The volume of a spherical *sector* (which is formed by a cap and a cone, with the same base, whose vertex is at the centre of the sphere) is got by multiplying the area of the cap by one-third of the radius of the sphere. If h is the height of the cap

$$\therefore \text{ Volume of sector} = 2\pi rh \times \tfrac{1}{3}r = \tfrac{2}{3}\pi r^2 h.$$

The volume of the whole *sphere* is deduced from this.

$$\therefore \text{ volume of sphere} = \tfrac{4}{3}\pi r^3 = \tfrac{1}{6}\pi d^3.$$

The volume of a *segment* (bounded by a cap and its plane base) is found as the difference between the volumes of a sector and its cone.

The volume of a *zone* is found as the difference between two segments.

The volumes of a cone, pyramid, frustum of cone or pyramid, sphere, spherical segment or zone can all be found from the *Prismoid Formula*,* $\mathbf{V} = \frac{1}{6}(\mathbf{T} + 4\mathbf{M} + \mathbf{B})\boldsymbol{h}$ where T,B are the areas of the top and base, M is the area of a section half way up, and $\boldsymbol{h}$ is the vertical height. This formula is true also for solids of the shapes shown in **Figs. 70** and **73**.

Examples—41 : Miscellaneous Solids

1. I wish to know the size of the Great Pyramid, but one book gives only the vertical height as 481 ft., and another gives the slope of the faces as 51°51′. Calculate the side of the base, which is square, and the slope of an edge.

2. Find the area of the curved surface of a cone whose vertical height is 10 ft. and vertical semi-angle 20°.

---

* **Or Simpson's Rule.**

3. Find the surface-area of a silk lampshade in the form of a frustum of a cone, the radii being 5 in. and 8 in., and the slant height making 70° with the base.

4. Find the volume of the cone in Ex. 2.

5. Taking the Earth's radius as 3960 miles, find the area within the Arctic circle, which is the parallel of latitude 66°33′N. [This is a spherical cap.]

6. Find the area of the zone of the Earth's surface between parallels of latitude 41° N. and 45° N. The American state of Wyoming is between these parallels, and its two other boundaries are the meridians 27° W. and 34° W. Calculate its area.

7. A soap-film is formed across the mouth of a funnel whose diameter is 10 cm., and a bubble is blown of diameter 20 cm. In what ratio is the bubble thinner than the original flat film ? [Compare their areas.]

8. A sphere of iron of diameter 2 in. has a conical hole bored in it, the vertex of the cone being at the centre of the sphere. If the vertical angle of the cone is 50°, calculate the volume removed. [The shape of the metal removed is a spherical sector.]

9. Find the volume of glass in a lens with one surface plane and of diameter 6 cm. and the other convex and part of a sphere of radius 5 cm. [This is a spherical segment, *i.e.*, a sector minus a cone, Fig. 97. Calculate (i) MC, (ii) NM, (iii) area of cap ANB, (iv) vol. of sector ANBC, (v) vol. of cone ABC.]

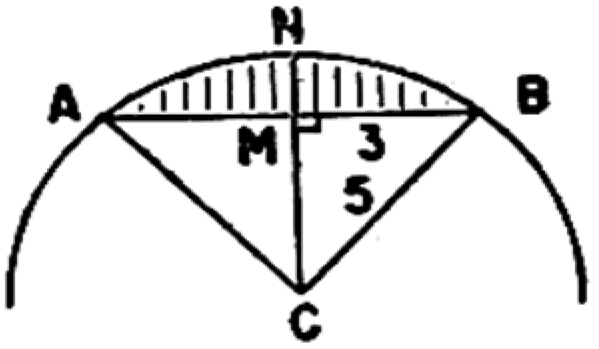

FIG. 97.

10. A wooden ball of diameter 6 in. floats in water so that its highest point is 2 in. above the water surface. Calculate the volume immersed.

11. A pail has the shape of a frustum of a cone, its vertical height being 10 in. and its top and bottom diameters 14 in. and 8 in. Calculate its capacity in gallons (taking 1 gallon ≡ 277·8 cu. in.). [Use the prismoid rule, the section half-way up being clearly a circle of diameter 11 in.]

12. A slag heap has the shape shown in Fig. 73, its base being a rectangle 50 yd. by 120 yd., its height being 60 ft. and the slope of the sides 40° to the horizontal. Find its volume by the prismoid rule.

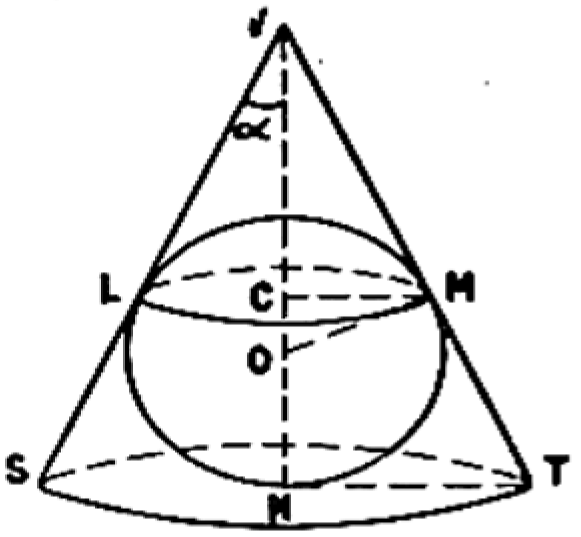

FIG. 98.

13. Fig. 98 shows a sphere of radius r inscribed in a cone whose vertical semi-angle is α. Express in terms of r and α the lengths of VM, CM, VO, CO, VN, NT.

14. If OM = 3 in. and OV = 5 in., show that the area of the surface of the sphere is equal to that of the base of the cone (Fig. 98).

15. Find the volume of the space inside the cone and above the sphere in Ex. 14.

REVISION EXERCISES

VA

1. A beam of parallel light of width x strikes a wall h feet away at right angles. If the beam is turned through an angle $\theta°$, what are (i) the length of the beam, (ii) the width of the patch of light on the wall. (Fig. 99.)

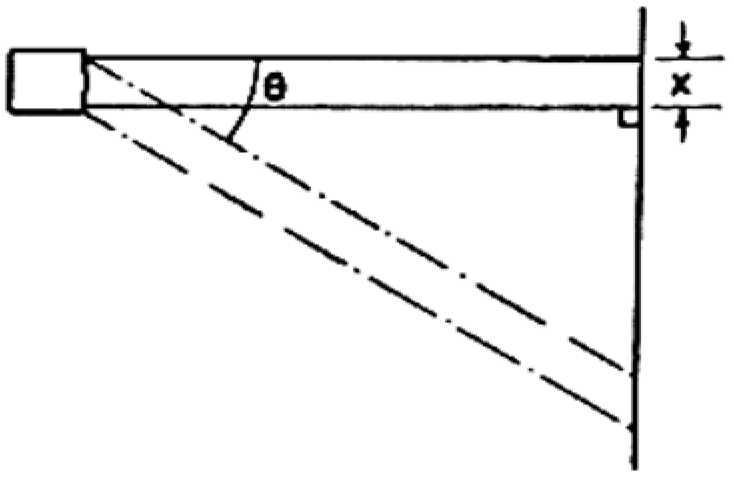

FIG. 99.

2. A four-sided figure ABCD has AB = 8″, BC = 7″=DA, CD = 4″ and AC = 9″. Show that AB is parallel to DC. [Prove cos BAC = cos ACD.]

3. A pyramid VABCD has a rectangular base ABCD with VA=VB=VC=VD=11″, AB=CD″=4, BC=DA=12″. Calculate (i) the vertical height, (ii) the angle between VA and the horizontal, (iii) (iv) the angles between the faces VAB, VBC and the horizontal.

4. ABC is a triangle with AB = AC = 10″. AD meets BC at D, and ∠BAD = 40°, ∠DAC = 20°. Calculate AD, BD, CD.

VB

1. A ball rests inside a flower pot, touching the base and the "sides." The diameters of the top and bottom of the pot are 10″ and 8″, and the slant height is 12″. Find the radius of the ball. [Find the angle at which the slant height meets the base.]

2. A triangle ABC has AB = 8″, BC = 7″, CA = 6″. Prove that cos B = 11/16. If T is a point on BC such that BT = 4″, calculate AT. Show that AT bisects the ∠BAC

by proving cos BAT = cos CAT. Of what geometrical theorem is this a special case ?

3. AB is the ridge of a roof whose ends APS, BQR are vertical and whose sides ABQP, ABRS are rectangles. AP = AS = 20 ft., AB = 48 ft., and PS = QR = 16 ft. Calculate (i) the height of AB above PQRS, (ii) the length of AR, (iii) the angle between AR and the horizontal, (iv) the angle QAR.

4. Fig. 100 shows some contours on a map whose scale is 1 cm. for 100 yd.

(i) If, on the map, AB = 1·05 cm., BC = 1·40 cm., find the average slopes of AB and BC.

(ii) The dotted line represents a path ascending at a constant slope of $10°$ from A to D. Show that its length is about 384 yd., while the horizontal distance is about 378 yd.

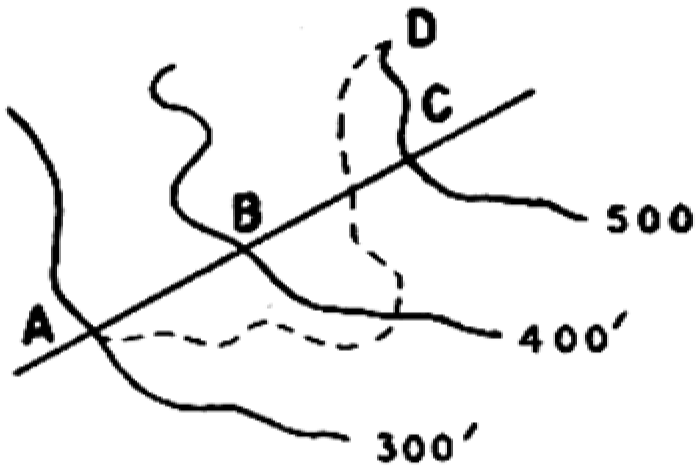

FIG. 100.

Vc

1. ABC is a triangle with $\angle B = 70°$, $\angle C = 50°$, BC = 10 in. M is the mid-pt. of BC, and P the projection of A on BC. If $\angle AMP = \theta$, what lengths on the figure represent AP cot B, AP cot C, AP cot θ ? Show that 2MP = PC — PB and deduce $2 \cot \theta = \cot C - \cot B$. Hence find θ and AM.

2. ABCD is a quadrilateral with AB = 11 in., BC = 12 in., CD = 8 in., DA = 6 in., and AC = 12 in. Show that it is cyclic by calculating cos B and cos D.

3. In triangle ABC, right-angled at B, P is the projection of B on AC and Q the projection of P on AB. If PQ = r and $\angle ABP = \beta$, express BP, BC, CA in terms of r and β.

4. A pyramid has a square base and each pair of *opposite* slant edges meet at right angles. Show that *adjacent* slant edges meet at $60°$ and that the slope of a slant face is $\sec^{-1}\sqrt{3}$. ($54°44'$.)

Vd

1. A roof, whose shape is that of Fig. 70, has its horizontal base 54 ft. by 22 ft. and its sides slope at $50°$ to the horizontal. Calculate the total area of the four sides.

2. If $\sin \alpha = 1\frac{1}{2} \sin \beta$, find the ratio $\tan \alpha : \tan \beta$ (i) if $\beta = 5°$, (ii) if $\beta = 30°$.

3. ABC is a triangle with a right angle at C. D is a point on BC such that BD = DA. By finding lengths in the figure equal to b cot B, b cot 2B and b cosec 2B, prove that cosec 2B = cot B − cot 2B. Also, if AD = d, find lengths in the figure equal to 2d cos B, d sin 2B, and deduce that sin 2B = 2 sin B cos B.

4. From a ship a landmark bears N.25°W. After the ship has sailed for 10 miles on a course N.75°E. the landmark bears N.73°W. Find the distances of the ship from the landmark at each observation.

VE

1. ABCD is a square of side l. A line DEF cuts CB at E and meets AB, produced, at F, and angle CDE = θ. Express EF in terms of l and θ.

2. A tripod made of thin iron rod has a triangular top whose sides are all 6 in. long. A spherical glass bulb of diameter 6 in. is supported on the tripod. How far is the bottom of the bulb below the level of the top of the tripod? [Consider a section of the sphere by the plane of the tripod.]

3. A,B,C,D are four points on a proposed railway track. The angles ABC, BCD are both 145°, and BC = 880 yds. The parts AB, CD are to be straight, and B is to be joined to C by an arc of a circle. What is the radius of this arc? How many 40-ft. rails will be needed for a single track from B to C? [AB and CD must be tangents to the circle.]

4. A balloon is observed to be due north at elevation 11° from an observation post A and to be at elevation 10° from B, another O.P., which is 10,000 yards due east of A. Find the height of the balloon and its distance from A.

CHAPTER VI

MORE TRIANGLE FORMULÆ

The Formula $\triangle = \sqrt{\{\mathbf{s(s-a)(s-b)(s-c)}\}}$

THIS formula expresses $\triangle$, the area of a triangle, in terms of the lengths a, b, c of its sides. s is used to mean "semi-sum-of-sides," *i.e.*, $s = \frac{1}{2}(a + b + c)$.

This usage is well established and must be known.

The formula for $\triangle$ not only gives the area of the triangle, but occurs in various other important formulæ for lengths connected with the triangle, some of which are used in solving for unknown sides and angles.

It is probably best to practise using the formula before learning the proof, which is not very easy; but if you are prejudiced against using an unproved formula, the set of examples below can be postponed.

The formula is easy to remember and should be learnt by heart. It is known as Hero's or Heron's formula, and is at least 2000 years old.

Examples—42

Using $\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}$

1. For the triangle in which $a = 3$ in., $b = 4$ in., $c = 5$ in. find the values of s, $s - a$, $s - b$, $s - c$; calculate $\triangle$ and verify otherwise.

2. Repeat Ex. 1 for the triangle whose sides are 5 in., 12 in., 13 in. and that whose sides are 8 in., 15 in., 17 in.

3. Show that the area of an equilateral triangle of side 2 in. is $\sqrt{3}$ sq. in. (i) using the formula, (ii) otherwise.

4. Find from the formula and check otherwise the areas of triangles whose sides are (i) 13 cm., 13 cm., 10 cm.; (ii) 8 cm., 8 cm., 6 cm.

5. Using logarithms, find the area of the triangles given :—

(i) $a = 42{\cdot}3$ cm., $b = 34{\cdot}8$ cm., $c = 30{\cdot}1$ cm. (ii) $a = 193'$, $b = 194'$, $c = 195'$. (iii) $a = 19{\cdot}74$ in., $b = 17{\cdot}72$ in., $c = 15{\cdot}88$ in. (iv) $a = 14{\cdot}7$, $b = 17{\cdot}1$, $c = 10{\cdot}3$.

			Logs
[Sol'n of (i)	$a = 42{\cdot}3$	$s - a = 11{\cdot}3$	1·0531
	$b = 34{\cdot}8$	$s - b = 18{\cdot}8$	1·2742
	$c = \underline{30{\cdot}1}$	$s - c = \underline{23{\cdot}5}$	1·3711
	$2s = 107{\cdot}2$	sum $= 53{\cdot}6 = s$	$\underline{1{\cdot}7292}$
	$s = 53{\cdot}6$	(see No. 3, p. 91)	$2\overline{)5{\cdot}4276}$
	$\triangle = 517{\cdot}4$ sq. cm.		2·7138

The arrangement in columns as shown is recommended.]

Methods of Proof

The formula can be proved in two completely different ways :—

(1) By the use of the figure of a triangle showing the inscribed and an escribed circle. s, $s - a$, $s - b$, $s - c$ are shown as lengths in this figure.

(2) By the use of the figure of a triangle with one altitude drawn. This method uses Pythagoras' theorem and involves fairly elaborate work with factors.

These methods will now be given in turn, as sets of examples.

Examples—43

The lengths s, s — a, s — b, s — c

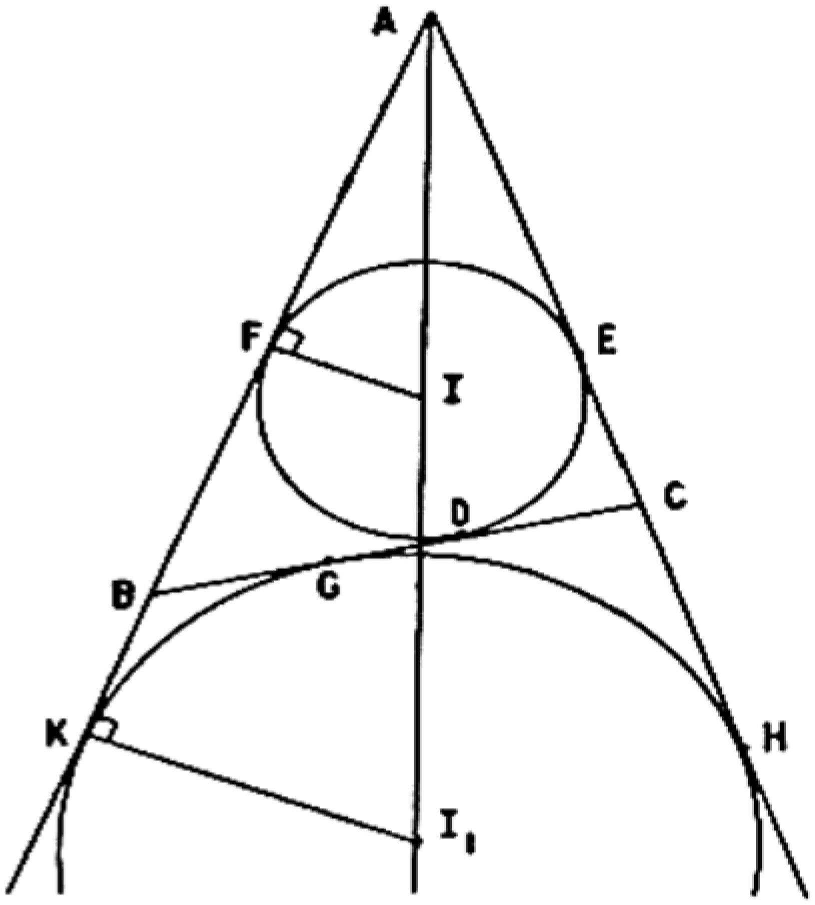

FIG. 101.

1. In Fig. 101 take x, y, z for the tangents from A, B, C to the inscribed circle, *i.e.*, let AF $= x$, BD $= y$, CE $= z$.

Show in succession that

$$2x + 2y + 2z = a + b + c = 2s$$
$$x + y + z = s$$

but $y + z = a$

$$\therefore\ x = s - a.$$

What are y and z?

2. If AK and AH are tangents to the escribed circle shown, prove

BK + CH $= a$,

AK + AH $= a + b + c = 2s$

$\therefore$ AK = AH $= s$.

Find lengths of BG, CG and show BG = CD.

3. *Prove that* $\mathbf{s - a + s - b + s - c = s}$

[This important result should always be used as a check when these four lengths are found from given values of a, b, c.]

4. Prove that $r = (s - a) \tan \frac{1}{2}A$ and give two corresponding results.

5. Prove that $r_1 = s \tan \frac{1}{2}A$; prove also that

$$r_1 = (s - c) \cot \tfrac{1}{2}B = (s - b) \cot \tfrac{1}{2}C.$$

6. Find the lengths of all the segments of the sides of the triangle in Fig. 101 when $a = 13$, $b = 14$, $c = 15$.

[AF, FB, BK, BG, BD, GC, DC, AE, EC, CH.]

Verify that $AK = s = AH$.

7. In order to find the diameter of a gasometer, three upright posts are set up at A, B, C so that each can just be seen from the other two past the side of the gasometer. [Fig. 101 will serve.]

If AB = 87 ft., BC = 97 ft., CA = 92 ft., find $\angle A$ by means of the cosine formula ; the result of No. 4 may then be used to find the required diameter.

The Formula $\triangle = rs = r_1 (s - a)$:

Radii of Inscribed and Escribed Circles

The first two formulæ are important in themselves and also lead to the proof of the formula for $\triangle$ in terms of the sides.

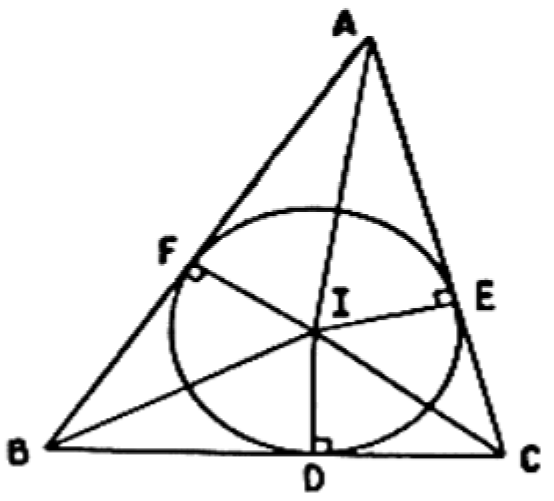

FIG. 102.

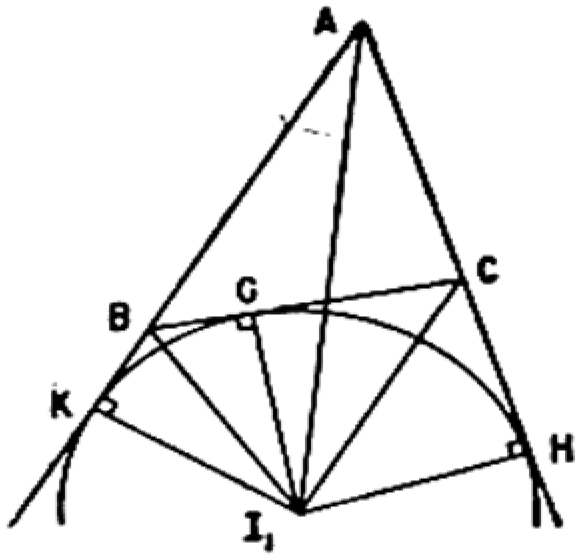

FIG. 103.

In Fig. 102 $\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB$.

Why is $\triangle IBC = \frac{1}{2}ar$ (where r is the radius of the inscribed circle) ?

Deduce that $\triangle ABC$ that is (for short) $\triangle = rs$.

In Fig. 103 $\triangle ABC = \triangle I_1CA + \triangle I_1AB - \triangle I_1BC$.

Why is $\triangle I_1CA = \frac{1}{2}br_1$ (where r_1 is the radius of the escribed circle) ? Deduce that $\triangle ABC = r_1(s-a)$.

Examples—44

1. Write out more fully the two proofs above, drawing your own figures.

2. What are the other two formulæ corresponding to $\triangle = r_1(s - a)$?

3. Writing the formulæ $r = \dfrac{\triangle}{s}$, $r_1 = \dfrac{\triangle}{s - a}$, show that the radii of the inscribed and the three escribed circles of the triangle with sides 3 in., 4 in., 5 in. are 1 in., 2 in., 3 in., 6 in.

4. Show that the radius of the inscribed circle of the triangle with sides 5, 12, 13 cm. is 2 cm. and find the radii of the escribed circles.

Proof of $\triangle = \sqrt{\{s(s - a)(s - b)(s - c)\}}$

[Work the three following examples as a proof.]

5. Prove that $\triangle^2 = rr_1\, s(s - a)$ and write down two similar formulæ. To obtain the usual formula for $\triangle^2$ to what must rr_1 be proved equal ?

6. In Fig. 102 prove $r = (s - b) \tan \frac{1}{2}B$.

In Fig. 103 prove $\angle BIG = \frac{1}{2}B$ (find $\angle IBG$ first) and deduce that $r_1 = (s - c) \cot \frac{1}{2}B$.

7. From Ex. 6 deduce that $rr_1 = (s - b)(s - c)$ and hence from Ex. 5 deduce that $\triangle^2 = s(s - a)(s - b)(s - c)$.

Algebraic Proof for $\triangle$

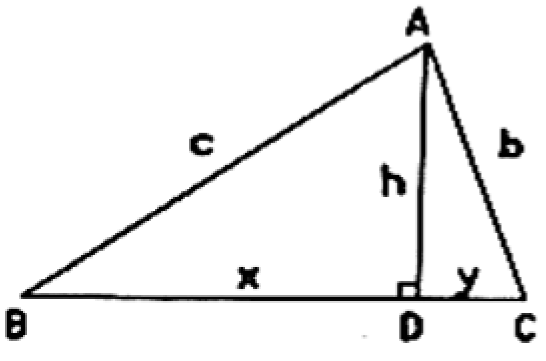

FIG. 104.

With the notation of Fig. 104, in which AD is perp. to BC,

$$x + y = a$$

$$\text{and } x^2 - y^2 = c^2 - b^2 \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore x - y = (c^2 - b^2)/a$$

$$\text{hence } 2x = a + \frac{c^2 - b^2}{a} \quad \therefore x = \frac{c^2 + a^2 - b^2}{2a} \quad . \quad (2)$$

$$\therefore h^2 = c^2 - x^2 = (c + x)(c - x)$$

i.e.,
$$h^2 = \left\{c + \frac{c^2 + a^2 - b^2}{2a}\right\} \left\{c - \frac{c^2 + a^2 - b^2}{2a}\right\} \quad (3)$$

Examples—45

[Work straight through Nos. 1 to 4.]

1. Prove the statement (1) above by applying Pythag. to $\triangle$s ABD, ACD.

2. Show that the statement (2) above gives the value of cos B in terms of the sides.

3. Show that the first bracket on the r.h.s. of (3) may be written $\frac{1}{2}\{(a+c)^2 - b^2\}/a$ and factorise this. Show also that the second bracket $= \frac{1}{2}\{b^2 - (a-c)^2\}/a$ and factorise this expression. If examples 1, 3 have been worked it will be seen that

$$4a^2h^2 = (a+b+c)\,(a+c-b)\,(b+a-c)\,(b+c-a).$$

4. Show that $\triangle^2 = \frac{1}{4}a^2h^2$

$$= \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{c+a-b}{2} \cdot \frac{a+b-c}{2}$$

$$= s(s-a)\,(s-b)\,(s-c).$$

5. Discuss any change needed in the above argument if $\angle$C is obtuse and show that the result is unaltered.

6. Show that the formula for $\triangle^2$ if multiplied out, instead of being factorised, comes to

$$\{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4\}/16.$$

This proof is both shorter and more elementary than the other but it needs some skill with factors, and as it does not exhibit s, $s-a$, etc., as lengths in the figure, it does not suggest, as the other does, why these expressions occur in the answer.

Use of the Formula for $\triangle$ in Solution of Triangles

Since $\triangle = \frac{1}{2}bc \sin A$ (being $\frac{1}{2}$ base × altitude, see Fig. 105) it follows that $\sin A = \dfrac{2\triangle}{bc}$.

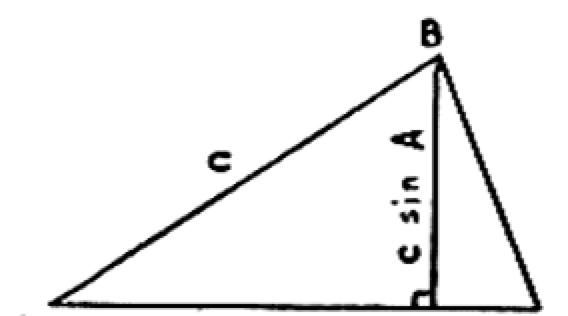

FIG. 105.

Hence in the S.S.S. case of solution, we can find $\triangle$ as $\sqrt{s(s-a)\,(s-b)\,(s-c)}$ and then sin A from the above.

Thus in Ex. 5 (i) worked on p. 90, $\sin A = \dfrac{2 \times 517{\cdot}4}{34{\cdot}8 \times 30{\cdot}1}$.

Examples—46

1. Work out the value of A from the formula for sin A just given. Write down and work out corresponding formulæ giving B and C. Check the value of A by angle-sum.

2. Use the formulæ sin A $= 2\triangle/bc$, etc., to find the angles if the sides are (i) 3 in., 4 in., 5 in.; (ii) 32 ft., 41 ft., 27 ft.

3. What difficulty arises if the triangle may be obtuse angled? How can it be avoided?

[The above method for solution in the S.S.S. case is probably preferable to the use of the Cosine Formula, but it is not stressed since it is inferior to the following *which is recommended as the best method.*]

Use of the Formula for *r* in Solution of Triangles

Since $r = \dfrac{\triangle}{s}$ and $\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}$

we get

$$r^2 = \frac{\triangle^2}{s^2} = \frac{s(s-a)(s-b)(s-c)}{s^2} = \frac{(s-a)(s-b)(s-c)}{s}$$

$$\therefore\ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad . \quad . \quad . \quad (1)$$

Further, since $r = (s-a)\tan\frac{1}{2}A$, we get

$$\tan\tfrac{1}{2}A = r/(s-a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These formulæ (1) and (2) should be used in the S.S.S. case of solution of triangles to find ½A, ½B *and* ½C, so that a check is obtained from the angle-sum. These formulæ are very suitable for logarithmic work, and the arrangement shown on the next page is recommended.

If only one angle is required it may be shorter not to find log *r* but to use

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

It may be of interest to try some one example by the various available methods and compare the amounts of work required. Also to test the accuracy likely in other cases by seeing how far the results of the various methods differ from each other.

Solve the triangle, if $a = 42{\cdot}6$ *ft.*, $b = 39{\cdot}4$ *ft.*, $c = 26{\cdot}8$ *ft.*

		Logs	Log Tan ½A etc	½A etc	A etc
$a = 42{\cdot}6$	$s - a = 11{\cdot}8$	1·0719	$\bar{1}{\cdot}9047$	38°46′	77°32′
$b = 39{\cdot}4$	$s - b = 15{\cdot}0$	1·1761	$\bar{1}{\cdot}8005$	32°17′	64°34′
$c = 26{\cdot}8$	$s - c = 27{\cdot}6$	1·4409	$\bar{1}{\cdot}5357$	18°57′	37°54′
$2s = 108{\cdot}8$		3·6889		90°0′	180°0′
$s = 54{\cdot}4$	$s = 54{\cdot}4$	1·7356			
		2)1·9533			
	$\log r =$	0·9766.			

The check of the second column should not be omitted.

The subtractions giving the second and fourth columns are helped by writing 54·4 and 0·9766 at the edge of a scrap of paper and holding this above the number to be subtracted.]

Examples—47

Use the above method to solve triangles whose sides are :—

1. 21·3, 30·4, 33·7 ft.
2. 6·31, 7·53, 9·88 in.
3. 8·33, 6·59, 12·54 ft.
4. 393, 659, 788 m.
5. 15·1, 26·3, 33·0 cm.
6. 1270, 1184, 486 yds.
7. Given $a = 15$, $b = 17$, $c = 20$, find C by using

$$\text{(i) } \cos C = \frac{a^2 + b^2 - c^2}{2ab}. \qquad \text{(ii) } \sin C = \frac{2\triangle}{ab}.$$

$$\text{(iii) } \tan \tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Dimensions in the *s*-formula

In the formula $\triangle = \sqrt{\{s(s-a)(s-b)(s-c)\}}$ the dimensions are correct, for an area is measured in square units, *e.g.*, sq. ft. or ft.^{2}, while the formula for $\triangle$ is the square root of the product of four lengths and $\sqrt{(\text{ft.}^4)} = \text{ft.}^2$

So also in $r = \sqrt{\{(s-a)(s-b)(s-c)/s\}}$ and

$$\tan \tfrac{1}{2}A = \sqrt{\{(s-b)(s-c)/s(s-a)\}}.$$

The first is $\sqrt{\{\text{ft.}^3/\text{ft.}\}} = \text{ft.}$, and the second is $\sqrt{\{\text{ft.}^2/\text{ft.}^2\}} = 1$.

SOLUTION OF TRIANGLES

The various cases have been discussed separately, in this and previous chapters. A summary is now given and suitable checks suggested.

(I) In the A.A.S. case use the sine formula.

Thus given A, B, c first find C from angle sum.

Then find a from $\frac{a}{\sin A} = \frac{c}{\sin C}$ and b from $\frac{b}{\sin B} = \frac{c}{\sin C}$.

To check find b again from $\frac{b}{\sin B} = \frac{a}{\sin A}$.

(II) In the A.S.S. case use the sine formula.

Thus given a, b, A find B from $\frac{\sin B}{b} = \frac{\sin A}{a}$.

Take both the acute and obtuse values of B (unless the angle-sum shows the obtuse value to be impossible) and in each case find C from angle-sum. Find c from $\frac{c}{\sin C} = \frac{a}{\sin A}$ in each case or else use $c = b \cos A + a \cos B$ to give *both* values of c.

To check, use $$\frac{c}{\sin C} = \frac{b}{\sin B}.$$

(III) In the S.A.S. case there are three methods which have been discussed :—

(i) Given b, c, A, find a from cosine formula, find both B and C by sine formula and use angle-sum as check.

Suitable if b and c are small whole numbers.

(ii) Given b, c, A, find C from the cotangent formula

$$\cot C + \cot A = \frac{b}{c \sin A}.$$

find a from $$\frac{a}{\sin A} = \frac{c}{\sin C},$$

find b from $\frac{b}{\sin B} = \frac{c}{\sin C}$; check from angle-sum.

(iii) Given b, c, A, find (B — C) from $\frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)} = \frac{b - c}{b + c}$:

hence find B and C ;

find a by sine formula ; check from $\frac{b}{\sin B} = \frac{c}{\sin C}$.

(IV) In the S.S.S. case again three methods have been mentioned :—

(i) Find the angles by the cosine formula, all three separately. Check by angle-sum. Suitable if the sides are not more than two figures.

(ii) Find $\triangle$ and use $\sin A = \frac{2\triangle}{bc}$ to find A and similarly B and C. Check by angle-sum.

(iii) Use the formula $\tan \frac{1}{2}A = \frac{r}{s - a}$, etc., to find each angle separately where $r = \sqrt{\{(s-a)(s-b)(s-c) \div s\}}$. Check by angle-sum.

This is the method recommended.

Distance between Two Observed Points

A fundamental problem in surveying is to find the distance between two observed points. This involves the solution of three triangles and care in checking is needed to ensure accuracy.

Examples—48

[All points named are coplanar.]

1. The base line AB is 425 yds. in Fig. 106. The angles at A and B are observed to be $\angle PAQ = 41°10'$, $\angle QAB = 56°38'$, $\angle PBA = 38°50'$, $\angle QBP = 32°5'$. The length and direction of PQ are to be found.

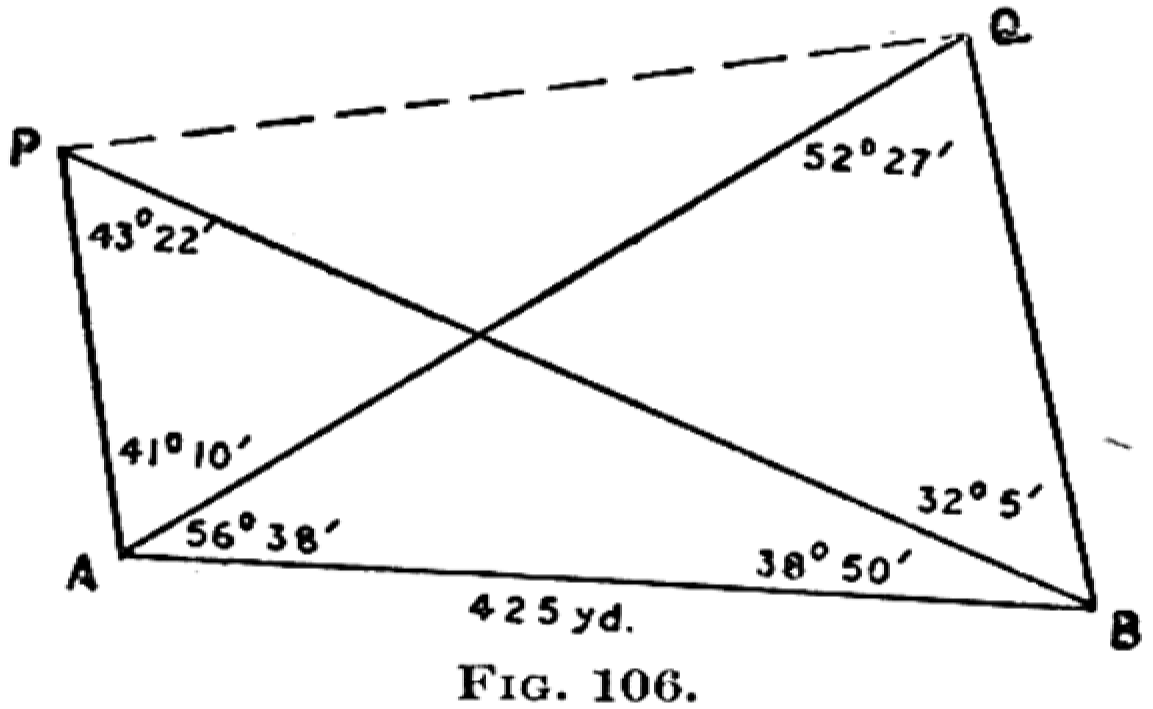

FIG. 106.

[Sol'n. $\triangle$s PAB, QAB are solved (case A.A.S.). Then PQ and its direction can be found from either $\triangle$PAQ or $\triangle$PBQ (case S.A.S.). By solving both the results are checked.]

$$\angle APB = 180^\circ - 41^\circ10' - 56^\circ38' - 38^\circ50' = 43^\circ22'$$
$$\angle AQB = 180^\circ - 56^\circ38' - 38^\circ50' - 32^\circ\,5' = 52^\circ27'$$

$$\therefore\ \frac{PA}{425} = \frac{\sin 38^\circ50'}{\sin 43^\circ22'} \qquad \begin{array}{r} 2\cdot6284 \\ \bar{1}\cdot8367 \\ \hline 2\cdot7917 \\ \bar{1}\cdot7973 \\ \hline 2\cdot5890 \end{array}$$

$PA = 388\cdot2$

$$\frac{PB}{425} = \frac{\sin 97^\circ48'}{\sin 43^\circ22'} \qquad \begin{array}{r} 2\cdot6284 \\ \bar{1}\cdot8367 \\ \hline 2\cdot7917 \\ \bar{1}\cdot9960 \\ \hline 2\cdot7877 \end{array}$$

$PB = 613\cdot4$

$$\frac{QA}{425} = \frac{\sin 70^\circ55'}{\sin 52^\circ27'} \qquad \begin{array}{r} 2\cdot6284 \\ \bar{1}\cdot8992 \\ \hline 2\cdot7292 \\ \bar{1}\cdot9754 \\ \hline 2\cdot7046 \end{array}$$

$QA = 506\cdot5$

$$\frac{QB}{425} = \frac{\sin 56^\circ38'}{\sin 52^\circ27'} \qquad \begin{array}{r} 2\cdot6284 \\ \bar{1}\cdot8992 \\ \hline 2\cdot7292 \\ \bar{1}\cdot9218 \\ \hline 2\cdot6510 \end{array}$$

$QB = 447\cdot7$

$$\text{In } \triangle PAQ,\ \frac{\tan \frac{1}{2}(P - Q)}{\tan 69^\circ25'} = \frac{QA - PA}{QA + PA} = \frac{118\cdot3}{894\cdot7} \qquad \begin{array}{r} \cdot4254 \\ 2\cdot0730 \\ \hline 2\cdot4984 \\ 2\cdot9516 \\ \hline \bar{1}\cdot5468 \end{array}$$

$\frac{1}{2}(P - Q) = 19^\circ24' \quad P = 88^\circ49' \quad Q = 50^\circ1'$

PQ makes with AB $88^\circ49' - 82^\circ12' = 6^\circ37'$

or $56^\circ38' - 50^\circ1' = 6^\circ37'$

$$\text{In } \triangle PBQ,\ \frac{\tan \frac{1}{2}(Q - P)}{\tan 73^\circ57\frac{1}{2}'} = \frac{PB - QB}{PB + QB} = \frac{165\cdot7}{1061\cdot1} \qquad \begin{array}{r} \cdot5414 \\ 2\cdot2193 \\ \hline 2\cdot7607 \\ 3\cdot0257 \\ \hline \bar{1}\cdot7350 \end{array}$$

$\frac{1}{2}(Q - P) = 28^\circ31' \quad Q = 102^\circ28\frac{1}{2}' \quad P = 45^\circ26\frac{1}{2}'$

PQ makes with AB $109^\circ5' - 102^\circ28\frac{1}{2}' = 6^\circ36\frac{1}{2}'$

or $45^\circ26\frac{1}{2}' - 38^\circ50' = 6^\circ36\frac{1}{2}'$

$$\frac{PQ}{PB} = \frac{\sin 32^\circ5'}{\sin 102^\circ28\frac{1}{2}'} = \frac{\sin 32^\circ5'}{\sin 77^\circ31\frac{1}{2}'} \qquad \begin{array}{r} 2\cdot7877 \\ \bar{1}\cdot7252 \\ \hline 2\cdot5129 \\ \bar{1}\cdot9897 \\ \hline 2\cdot5232 \end{array}$$

$PQ = 333\cdot6$ yds.

$$\text{or } \frac{PQ}{AQ} = \frac{\sin 41^\circ10'}{\sin 88^\circ49'} \qquad \begin{array}{r} 2\cdot7046 \\ \bar{1}\cdot8184 \\ \hline 2\cdot5230 \\ \bar{1}\cdot9999 \\ \hline 2\cdot5231 \end{array}$$

$PQ = 333\cdot5$ yds.

2. From two points A and B which are 69 yds. apart, the points P and Q are observed.

$$\text{Given } \angle BAP = 114^\circ \quad \angle BAQ = 62^\circ 15'$$
$$\angle ABP = 49^\circ 6' \quad \angle ABQ = 103^\circ 59'$$

Prove that $PQ \simeq 221$ yds. and find the angle which it makes with AB.

3. In the quadrilateral APQB it is found that

$$AB = 1246 \text{ ft.}, \angle BAP = 93^\circ 6', \angle BAQ = 44^\circ 48',$$
$$\angle ABP = 36^\circ 17', \angle ABQ = 59^\circ 9'.$$

Find the length of PQ and the angle PQA.

4. If in Fig. 107 any numerical values are assigned to a, b, c, d, e, then

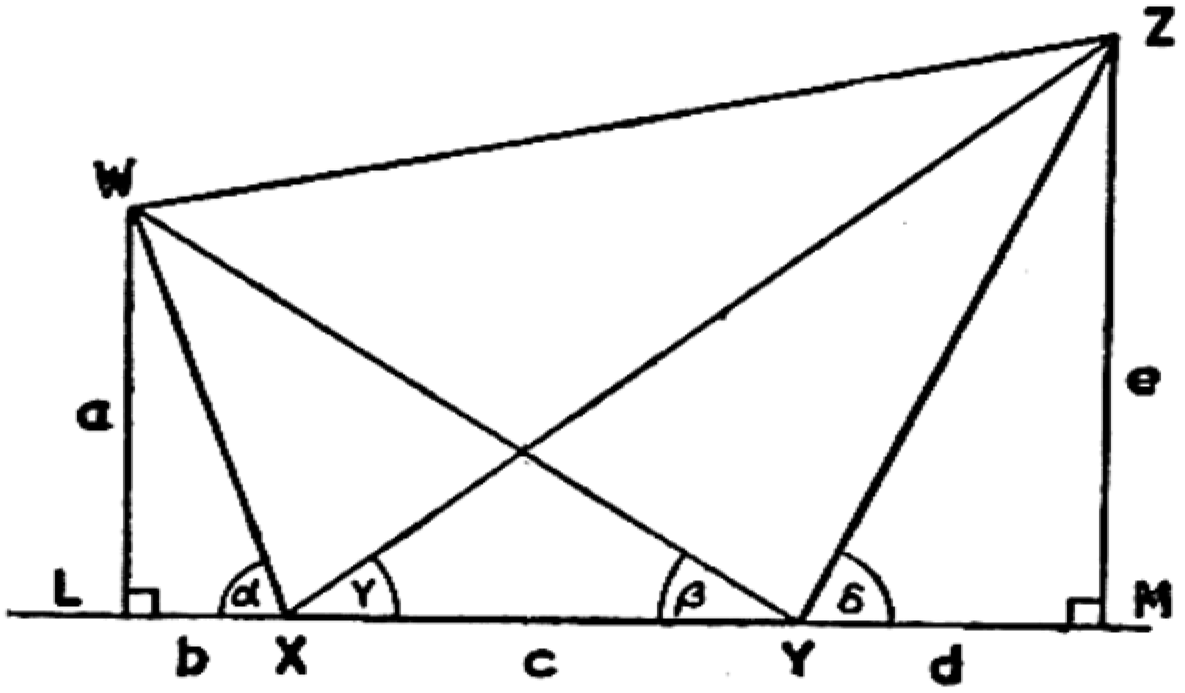

FIG. 107.

(i) $\alpha, \beta, \gamma, \delta$ can be found from $\tan \alpha = a/b$, $\tan \beta = a/(b + c)$, etc.

(ii) The length WZ can be found as

$$\sqrt{\{(e - a)^2 + (b + c + d)^2\}}.$$

(iii) The angle which WZ makes with LM can be found as $\tan^{-1} (e - a)/(b + c + d)$. Taking any numerical values you choose find answers to (i), (ii), (iii). Also using the value of c and the results of (i) solve the triangles WXY, ZXY, WXZ, WYZ and see if the results give the same length and slope for WZ as before.

AIR NAVIGATION

Course and Track Problems

The *course* of an aircraft is the angle which its longitudinal axis makes with the line to the north, or is the bearing of the direction in which it is *pointing*. This angle is measured clockwise *from* north as a three-figure group of degrees from 000 to 360 ; for example, N.27°E. is 027. The *track* is the path over the ground. It is different from the course except when there is no wind at all, or when the wind is dead ahead or astern.

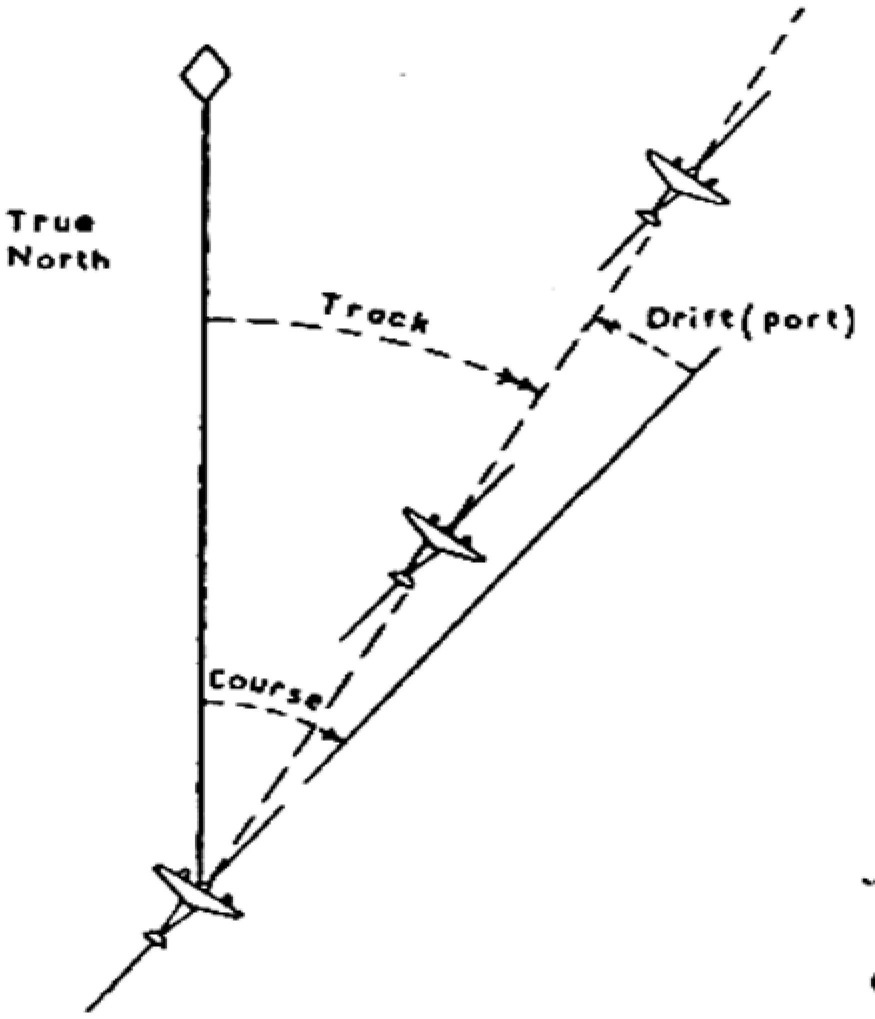

FIG. 108.

The *drift* is the angle *from* the axis to the track, and is reckoned starboard (right) or port (left), of the axis. It is the angle between the course and the track.

The air speed (A/S) is the speed of the aircraft through the air.

The ground speed (G/S) is the speed of the aircraft along the track.

The wind speed (W/S) is the speed of the wind, and the wind direction (W/D) is the direction *from* which the wind is blowing.

In the various problems which occur it is necessary to solve a triangle, either by drawing, by using a special slide-rule, or by trigonometry. The two first methods are used in practice, since great accuracy in the data is impossible, but it is important to understand the trigonometrical method and have it in reserve.

There are three main problems :—

(i) Given the required track, the wind, and the air speed, find the course and ground speed, *e.g.*,

Required track 275, W/D 340, W/S 40 m.p.h., A/S 110 m.p.h. AC is drawn along the required track. AB is drawn of length 40 for the wind. An arc BC of radius 110 from B is drawn for the air speed.

A rough sketch (Fig. 109) shows the course (direction of BC) as about 300 and G/S about 90. The case is A.S.S., but the solution is unique as BC > BA.

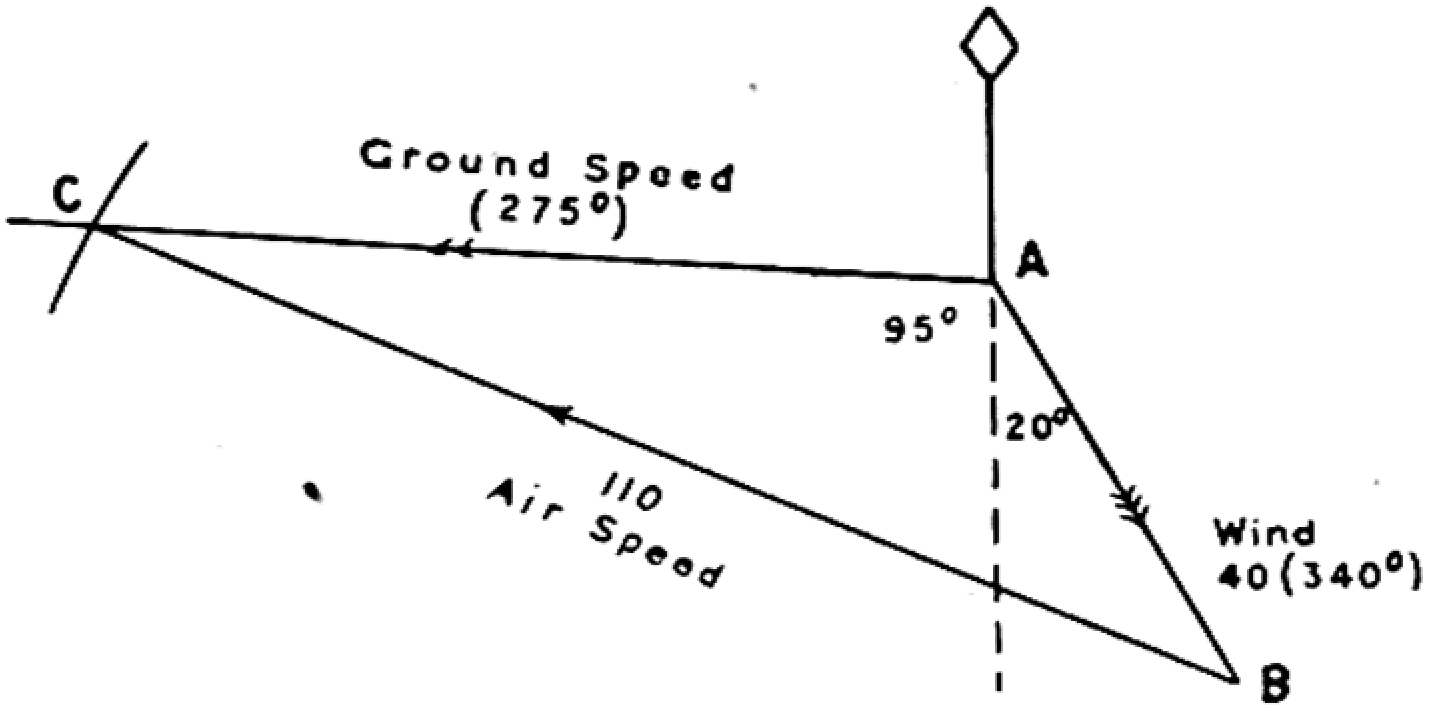

Fig. 109.

$$\sin C = \frac{40 \sin A}{110} \text{ and } A = 115^\circ$$

$$\therefore\ C = 19^\circ 15',\ B = 45^\circ 45'.$$

$\therefore$ Course is 294°15′ or 294°.

$$\text{Also } AC = \frac{110 \sin 45^\circ 45'.}{\sin 65^\circ}$$

$$= 86\cdot94.$$

$$\begin{array}{ll} \bar{1}\cdot9573 & 2\cdot0414 \\ 1\cdot6021 & \bar{1}\cdot8551 \\ \hline 1\cdot5594 & 1\cdot8965 \\ 2\cdot0414 & \bar{1}\cdot9573 \\ \hline \bar{1}\cdot5180 & 1\cdot9392 \end{array}$$

$\therefore$ Ground speed is 87 m.p.h.
The drift is 19° to port.

(ii) Given the course and air speed, find the track and ground speed. Air speed 200 m.p.h. Course 062; wind from 290; wind speed 50 m.p.h.; find track and ground speed.

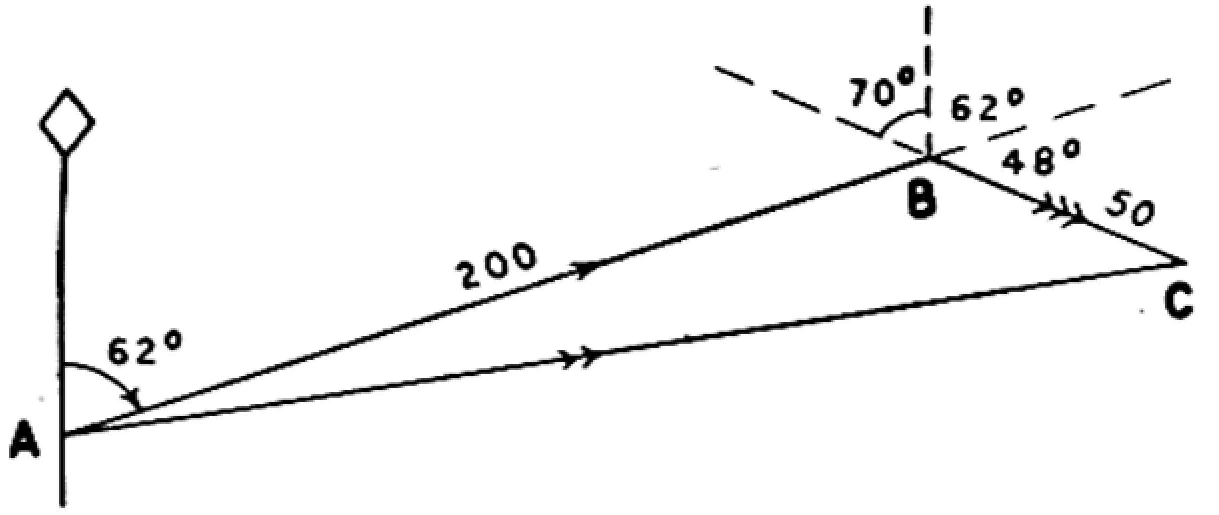

Fig. 110.

The diagram (Fig. 110) shows the track to be about 070 and the ground speed about 230 m.p.h. The case is the S.A.S. case of solution.

Using the cotangent formula $\cot A = 4 \operatorname{cosec} 48° + \cot 48°$

$$= 4 \times 1{\cdot}3456 + 0{\cdot}9004$$
$$= 6{\cdot}2828$$

$A = 9°3'$. This is the drift to starboard.

$$\begin{array}{r} 5{\cdot}3824 \\ 0{\cdot}9004 \\ \hline 6{\cdot}2828 \end{array}$$

The track is 071.

$$\text{Also } \frac{x}{50} = \frac{\sin 48°}{\sin 9°3'}$$
$$x = 236{\cdot}3$$

The ground speed is 236 m.p.h.

$$\begin{array}{r} 1{\cdot}6990 \\ \bar{1}{\cdot}8711 \\ \hline 1{\cdot}5701 \\ \bar{1}{\cdot}1966 \\ \hline 2{\cdot}3735 \end{array}$$

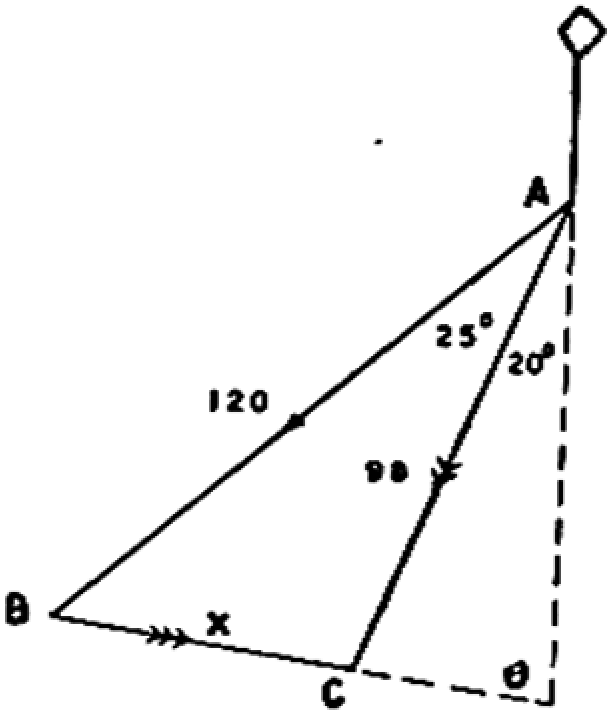

FIG. 111.

(iii) If the course, the track and the two corresponding speeds are known, the wind can be found in direction and speed.

Air speed 120, course 225, ground speed 98, track 200. This is the S.A.S. case of solution.

Using the tangent formula,

$$\tan \tfrac{1}{2}(C - B) = \frac{22}{218} \tan 77°30'$$
$$\tfrac{1}{2}(C - B) = 24°28'$$
$$C = 101°58'$$
$$\theta = 81°58'$$

$$\begin{array}{r} 1{\cdot}3424 \\ 2{\cdot}3385 \\ \hline \bar{1}{\cdot}0039 \\ 0{\cdot}6542 \\ \hline \bar{1}.6581 \end{array}$$

$$\frac{x}{\sin 25°} = \frac{120}{\sin 101°58'} = \frac{120}{\sin 78°2'}$$
$$x = 51{\cdot}83$$

$\therefore$ Wind speed $\simeq$ 52 m.p.h.

$$\begin{array}{r} 2{\cdot}0792 \\ \bar{1}{\cdot}6259 \\ \hline 1{\cdot}7051 \\ \bar{1}{\cdot}9905 \\ \hline 1{\cdot}7146 \end{array}$$

Examples—49

1. Given (i) track 306, A/S 150 m.p.h., W/D 063, W/S 51 m.p.h., find course and G/S.
   (ii) track 148½, A/S 132 m.p.h., W/D 029, W/S 56 m.p.h., find course and G/S.

2. Given (i) course 261, A/S 135 m.p.h., W/D 255, W/S 55·5 m.p.h., find track and G/S.
   (ii) course 060, A/S 200 m.p.h., W/D 020, W/S 50 m.p.h., find track and G/S.
3. Given (i) course 201, A/S 129 m.p.h., track 220, G/S 111 m.p.h., find W/D and W/S.
   (ii) course 280, A/S 200 m.p.h., track 270, G/S 190 m.p.h., find W/D and W/S.

The Observer-Battery-Target Problem

In Fig. 112 B is a battery, T its target. An observer at O measures $\angle$BOT and, with the range finder, OB and OT. He has to signal to the battery the angle OBT and the range BT. To find these is to solve $\triangle$OBT in the case S.A.S.

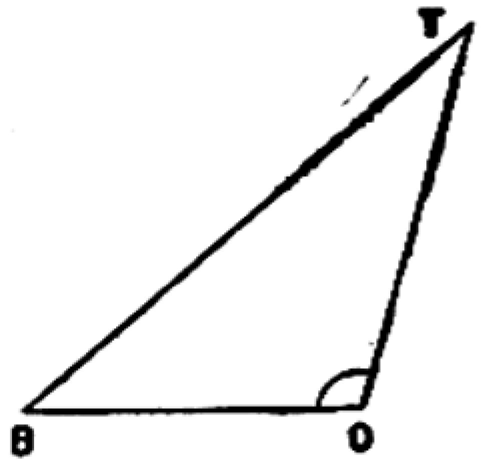

FIG. 112.

Examples—50

Find $\angle$OBT and BT from the following data :—

1. OB = 1520 yd., OT = 2110 yd., $\angle$BOT = 115°32′.
2. OB = 936 yd., OT = 3420 yd., $\angle$BOT = 65°14′.
3. OB = 1200 yd., OT = 1360 yd., $\angle$BOT = 87°42′.

Examples—51 : Miscellaneous

1. In Fig. 101, p. 90, prove the following results (i) FK $= a =$ EH ; (ii) GD $= b \sim c$; (iii) $\text{II}_1 = a \sec \frac{1}{2}\text{A}$; (iv) $\text{II}_1 = (r_1 - r) \operatorname{cosec} \frac{1}{2}\text{A}$.

From (iii) and (iv) deduce that $r_1 = r + a \tan \frac{1}{2}\text{A}$.

2. Calculate $\triangle$, r, r_1, r_2, r_3 for the triangle whose sides are 13, 14 and 15 in.

3. If A = 90°, prove that r, r_1, r_2, r_3 are equal respectively to $s - a$, s, $s - c$, $s - b$, and that $rr_1 = \triangle$.

4. ABC is a triangle with AC longer than AB. D is the foot of the perpendicular from A to BC. If BD $= y$, DC $= x$, prove that $c^2 - y^2 = b^2 - x^2$ and hence $x^2 - y^2 = b^2 - c^2$.

Deduce that $x - y = \dfrac{(b + c)(b - c)}{a}$.

Surveyors use this to solve a triangle in the S.S.S. case. Try it on the triangle in which $a = 20$, $b = 21$, $c = 19$. Since $x + y = a = 20$ we can find x and y separately and consequently cos B and cos C.

5. Solve the triangle in Qn. 4 (i) using the cosine formula, (ii) using the standard method [$\tan \frac{1}{2}A = r/(s - a)$]. (iii) using the formula $\sin A = 2\triangle/bc$, etc.

6. Solve the triangle $a = 102$, $b = 120$, $c = 43°20'$ by three methods.

7. Find the diameter of a gasometer by the " three posts " method of p. 91, Qn. 7, when AB = 115 ft., BC = 123 ft., CA = 110 ft. [Use the formula $r = \sqrt{\{(s-a)(s-b)(s-c)/s\}}$].

Aircraft Problems

8. An aircraft flying at 150 m.p.h. has to make good a track of 125. If the wind is 30 m.p.h. from 040 find the course to steer and the ground speed.

9. An aircraft has to make a journey of 217 miles on a track of 220, the air speed being 180 m.p.h. If the wind is 40 m.p.h. from 140 find the course to steer, the angle of drift, and the time taken.

10. An aircraft flies from A to B, a distance of 245 miles, in 1 hr. 45 min., the air speed being 160 m.p.h. The course set is 020 and the track 031. Find the direction and speed of the wind.

11. An aircraft flies on a course of 275 at 200 m.p.h. The angle of drift is 10° to port, and one mile is covered in 16½ sec. Find the direction and speed of the wind.

12. An aircraft flies on a course of 060 with air speed 200 m.p.h. The wind is 40 m.p.h. from 020. Find the track made good, the drift, and the ground speed.

13. Orders are given to an aircraft to leave base at 12·00 hrs. and reach a place 220 miles away on a bearing of 230 by 14·00 hrs. The wind is 35 m.p.h. from 130. Find the course to steer and the air speed required.

14. The distance from A to B is 200 miles, and an aircraft flies from A to B and back at an air speed of 180 m.p.h. The bearing of B from A is 097, and the wind is 40 m.p.h. from 160. Find the ground speed in each direction and the total time taken.

Practical Surveying Problems

15. AB is the vertical face of a tower standing on a slope TB. By means of a theodolite at T the angle of elevation (ATH) of A is found to be 37°45′.

By "levelling," the height BH of B above the horizontal through T is found to be 10·2 ft., and the length TB is found by means of a chain to be 141·3 ft. Find AB. (The height of the theodolite is ignored here.)

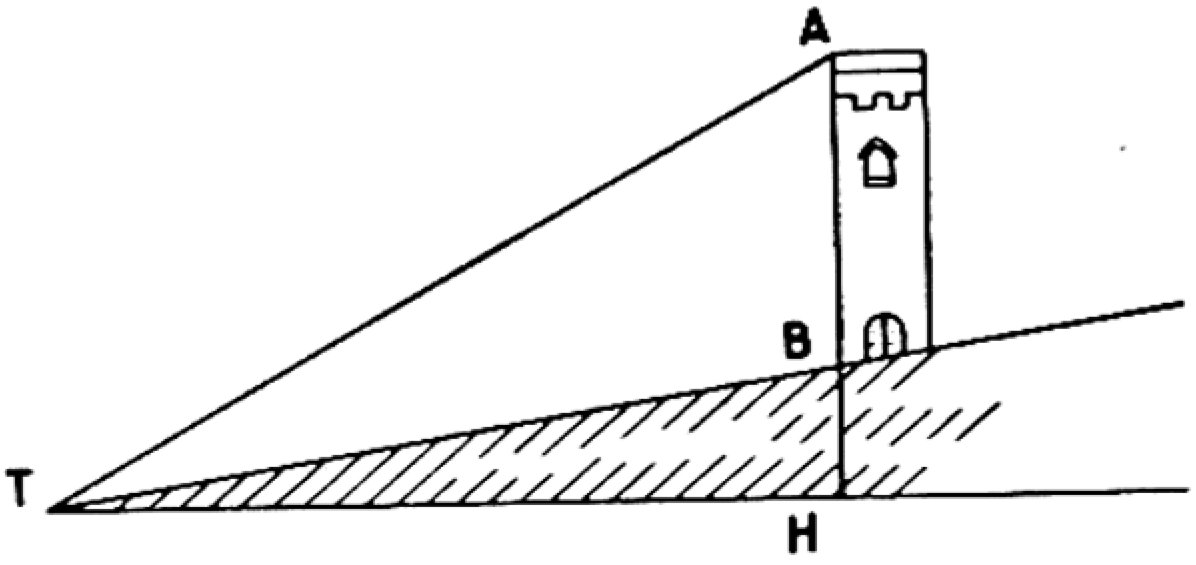

FIG. 113.

16. In Fig. 114, TG is the theodolite on its stand, AB a vertical line. Angle of elevation of A from T = 36°22′.

Distance GB = 138·4 ft.
Height HB = 9·9 ft.
Height of theodolite above ground, 5·2 ft.

Find the height AB.

[Find TL, which is parallel and equal to GH.]

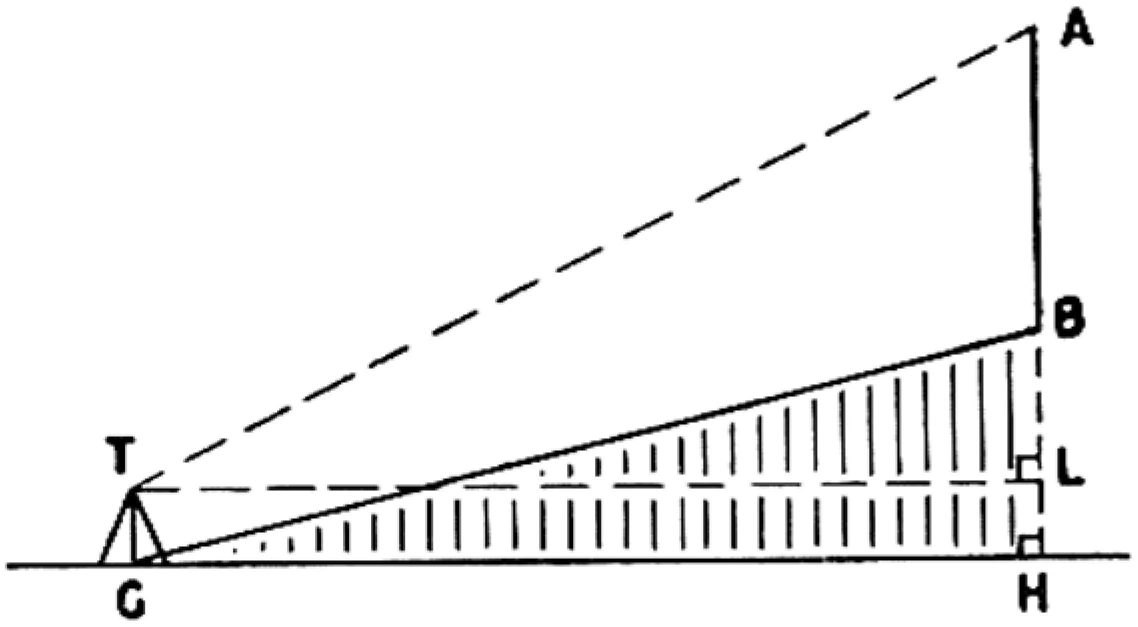

FIG. 114.

17. PQ is a wireless mast, P its top, observed from two points S and T which are such that QST is a horizontal line. ST = 123·6 ft., and the angles of elevation of P from S and T are 55°27′ and 40°33′. Find the height PQ.

18. In Fig. 115 the points ABST are in the same vertical plane. The straight distance ST is measured to be 212·7 ft.

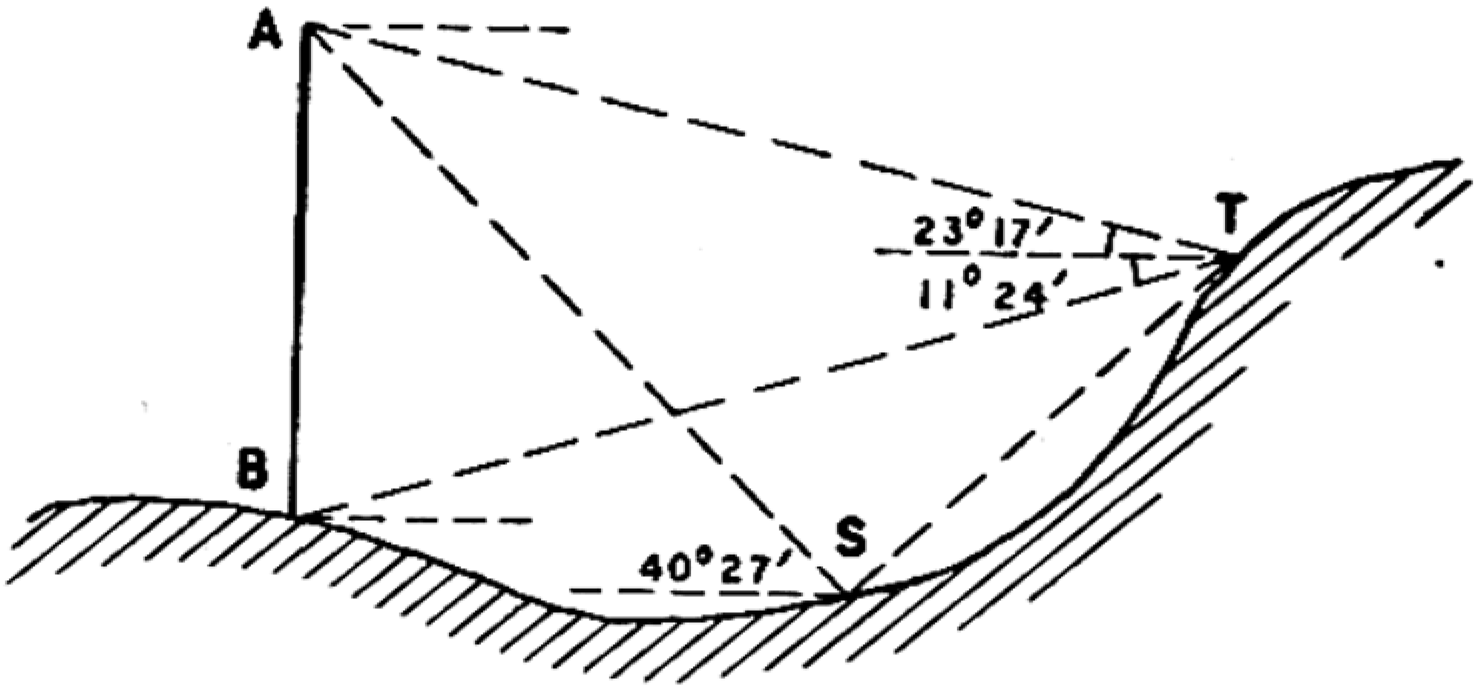

FIG. 115.

Angle of elevation of A from T, 23°17′.
,, ,, ,, ,, A ,, S, 40°27′.
,, ,, depression ,, B ,, T, 11°24′;
,, ,, ,, ,, S ,, T, 22°8′.

Show that $\angle SAT = 17^\circ 10'$ and find the (vertical) height AB. [Use sine rule.]

19. Suppose the points BST are in the same *horizontal* plane but not in the same straight line. ST = 192·4 ft. Angles of elevation of A from S and T are 33°41′ and 27°16′. Angle BST = 63°3′. Find the height AB.

[BS : BT = AB cot 33°41′ : AB cot 27°16′. Apply sine rule to △BST to get angle T; apply sine rule again to get BS.]

20. To find the length and angle of an inaccessible slope. The points ABST are in the same vertical plane. ST = 341·2 ft.

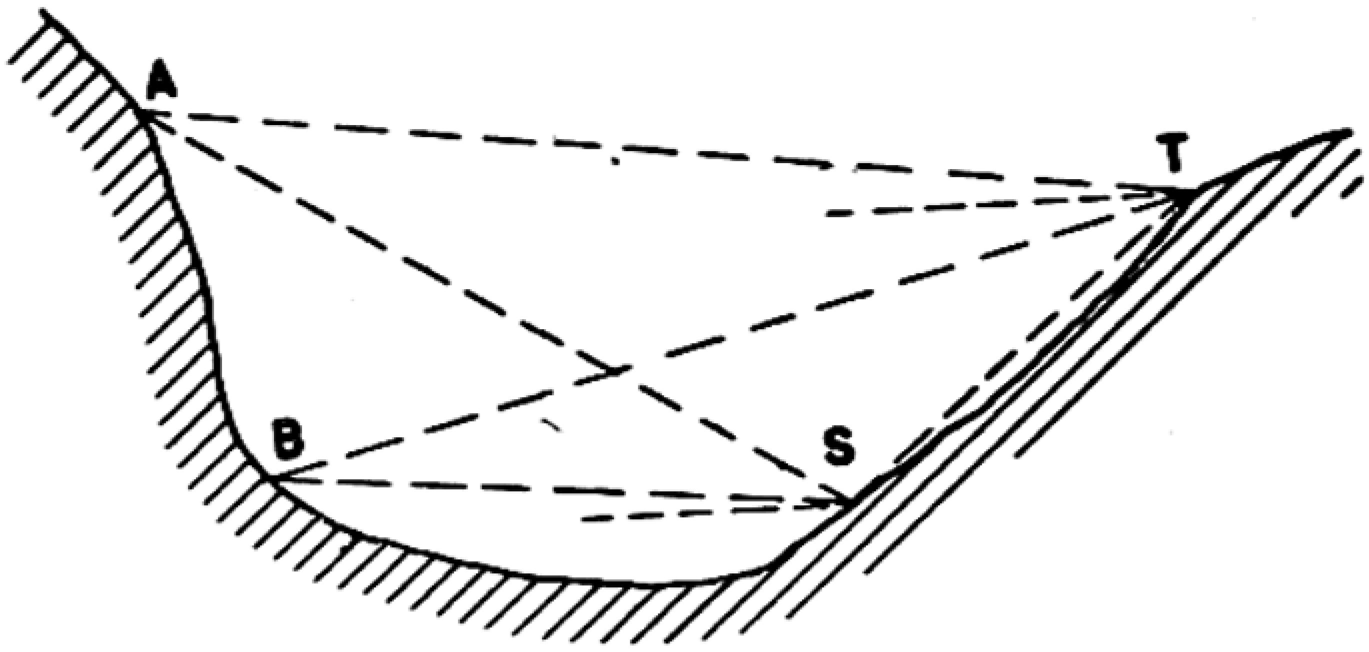

Fig. 116.

Elevation of A from T, 14°21′.
,, ,, A ,, S, 38°10′.
,, ,, B ,, S, 8°16′.
Depression ,, B ,, T, 11°22′.
,, ,, S ,, T, 25°10′.

Find the distance AB and the angle between AB and the horizontal.

CHAPTER VII

SMALL AND LARGE ANGLES. RADIAN MEASURE. THE ADDITION FORMULÆ

Small Angles

IF you compare the tables of (natural) sines and tangents for *small* angles you will see that there is not much difference between them. The smaller the angle, the more nearly equal are its sine and tangent, as this extract shows :

θ	60°	30°	10°	5°	1°	1° (using 7-figure tables)
sin θ	·8660	·5000	·1736	·0872	·0175	·0174524
tan θ	1·7321	·5774	·1763	·0875	·0175	·0174551

Four-figure tables show no difference between sin θ and tan θ if θ is 2° or less. Fig. 117 illustrates the relation between the sine and tangent of a small angle. AP is an arc of a circle whose centre is O. PN and TA are at right angles to OA. OA = OP = r, and AOP = θ.

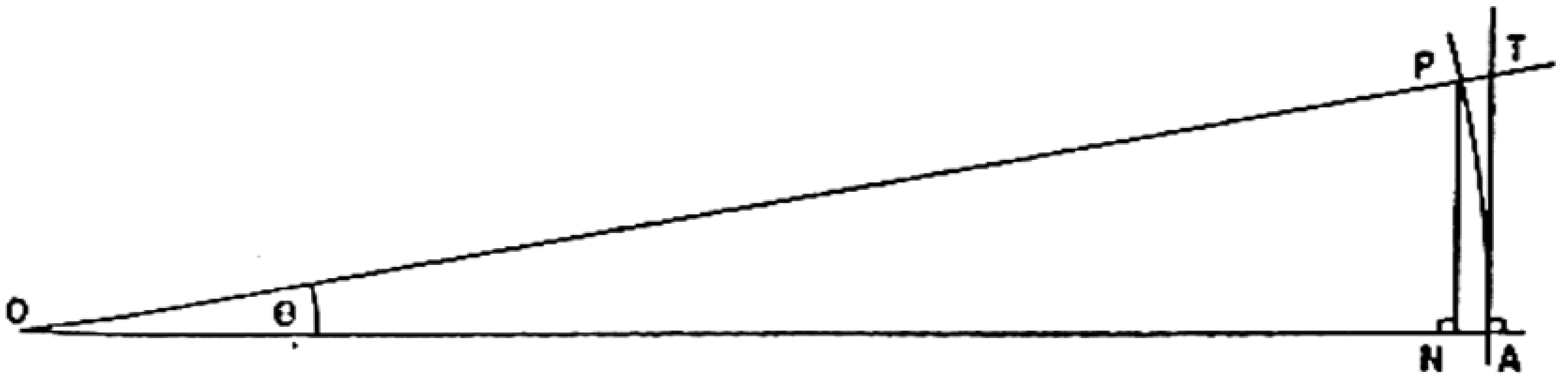

FIG. 117.

Now $\sin\theta = \frac{\text{PN}}{\text{OP}} = \frac{\text{PN}}{r}$, and $\tan\theta = \frac{\text{TA}}{\text{OA}} = \frac{\text{TA}}{r}$. But the lengths PN and TA are nearly equal, so that sin θ and tan θ are nearly equal.

Further, the figure suggests that the arc AP is intermediate in length between PN and TA, so that when these are nearly equal the arc may be taken as equal to either of them. This means that the sine or tangent of a small angle is very nearly equal to the arc divided by the radius, and so can be *calculated* as follows :—

The arc for a semicircle ($\theta = 180°$) is πr or $3{\cdot}1416\ r$.

,, ,, ,, an angle of 1° is $\frac{1}{180}$ of this or $\cdot 0175\ r$.

Hence arc/radius $= \cdot 0175 \simeq \sin 1° \simeq \tan 1°$.

In this way we can solve problems involving the sine or tangent of a small angle without the use of trigonometrical tables.

Examples—52

1. Using 3·1416 for π, calculate the arc of a circle of unit radius for an angle at the centre of (i) 5°, (ii) 10°. How much different is your second result from tan 10°?

2. For 15° show that the error in taking arc/radius for the tangent is about double that of taking arc/radius for the sine.

3. When the slope of a hill is given as 1 in x it sometimes means x along the slope and sometimes x along the horizontal. Show that this makes hardly any difference for 1 in 10, and find the difference in angle for a slope of 1 in 5.

4. Calculate $\sin 1\frac{1}{2}°$ and $\tan 2°$ to four decimal places.

5. Find the area of a triangle in which equal sides of 10 in. include an angle of 0·1°. (Use $\triangle = \frac{1}{2}ab \sin C$.)

Radian Measure

The calculation of the sine or tangent of a small angle requires the calculation of the ratio arc/radius. The value of this ratio depends on the angle only; if the radius is doubled so is the arc, if the angle remains the same. This gives us another way of measuring angles which is convenient for several purposes, and is known as "circular" measure or "radian" measure. The unit in this case is called a *radian*, and is the angle whose arc is equal to the radius. This angle is a little less than 60°,* and is, to the nearest minute, 57°18′.

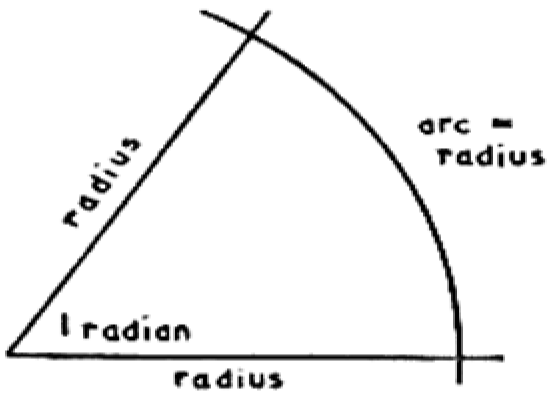

FIG. 118.

The arc for a semicircle is πr, so that the radian measure of 180° is π; in other words, 180 degrees $= \pi$ radians.

Using this relation, it is easy to express any angle in radians but it is more convenient to have a table of "Degrees and Radians."

---

* If the *chord* is equal to the radius the angle is 60°.

Advantages of Radian Measure

The formulæ for the length of an arc of a circle and for the area of a sector are much simpler if the angle at the centre is expressed in radians than if it is expressed in degrees; so much so that it pays to turn the degrees into radians first.

If l is the length of an arc subtending θ radians at the centre of a circle whose radius is r, then

$$l = r\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I)}$$

or arc = radius × radian measure of angle.

[The arc $= r \times \pi x/180$ if $x°$ is the angle subtended, but if $x° = \theta$ radians, then $\theta = \pi x/180$, since $180° = \pi$ radians.]

The area of a sector = radius × $\frac{1}{2}$ the arc $= r \times \frac{1}{2}r\theta$.

$$\therefore\ A = \tfrac{1}{2}r^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

[This replaces $A = \pi r^2 x/360$ if the angle is $x°$.]

The formulæ I, II and other formulæ derived from them are so much simpler than if the angle is measured in right-angles or degrees that radian measure is used in all theoretical mathematics. When a mathematician writes "an angle θ" he means θ radians and his abbreviation for a right-angle is $\frac{1}{2}\pi$.

Examples—53

Conversion

1. Working from $180° = \pi$ radians, find expressions such as $90° = \frac{1}{2}\pi$, for 60°, 30°, 45°, $22\frac{1}{2}°$, 15°, 120°, 135°, 270°, 720°.

2. Find the radian measure of 70° (i) approximately, as a fraction by taking π as $3\frac{1}{7}$, (ii) to four decimal places by taking π as 3·1416.

3. Show that, to the nearest minute, 1 radian = 3438 minutes. *Use this* to convert angles of 36°52′, 53°8′ to radians. Check from tables.

4. Express angles of $\frac{\pi}{5}, \frac{5\pi}{6}, \frac{7\pi}{6}$ radians in degrees, and evaluate $\sin\frac{\pi}{6}$, $\tan\frac{\pi}{4}$, $\sec\frac{\pi}{3}$, and with tables sin 1, sin 2.

5. Two angles of a triangle are $\frac{\pi}{4}$ and $\frac{\pi}{5}$ radians. What is the third?

Arcs. Sectors

6. A circle has radius 80 yds. What length of arc subtends an angle of 73°12′ at the centre ?

[Sol'n. Radian measure of $73°12' = 1{\cdot}2776$, from tables.

$$\begin{aligned}\text{Length of arc} &= \text{radian} \times \text{radian measure}\\ &= 80 \times 1{\cdot}2776 \text{ yds.}\\ &= 102{\cdot}1 \text{ yds.}]\end{aligned}$$

7. A circle has radius 100 yds. What length of arc subtends an angle of 43°36′ at the centre ?

8. What is the radius of a circle when an arc of length 50 ft. subtends an angle of 24° at the centre ?

9. What is the area of a sector of a circle whose radius is 10 in., the angle of the sector being (i) 2 radians, (ii) 53°8′ ?

10. What is the area of the sector whose arc and angle are given in No. 8 ?

Angular Velocity

11. The rate of rotation of a wheel or a shaft is usually in practice given in revolutions per minute ; but in theoretical work it is given in radians per second.

Show that (i) x revs. per min. $= \pi x/30$ radians per sec.

(ii) An angular velocity of ω (omega) radians per sec. means that a point on the rim of a wheel of radius r ft. is moving with an actual velocity v ft./sec. where $v = r\omega$.

12. A bicycle wheel of diameter 25 in. is lifted from the ground and spun with angular velocity 12 radians per sec. What is the speed of a point on the circumference in ft. per sec. ?

13. What is the angular velocity of the minute hand of a clock ? What is the speed of the tip of Big Ben's minute hand, which is 11 ft. long ?

14. What is the angular velocity of the earth in its rotation round its axis ? What are the actual speeds of a point (i) on the Equator, (ii) in latitude $51\frac{1}{2}°$ (London) ? Take the radius of the Earth as 3960 miles.

15. The speed of the tip of an airscrew must not exceed 900 ft./sec. If the diameter of the airscrew is 11 ft., find the maximum angular velocity and the maximum number of revolutions per minute.

Small Angles

16. A pole 6 ft. high is observed from a point at about the same level a quarter of a mile away. What is the angle subtended by the pole at the point?

[Sol'n. The pole can be regarded as an arc of a circle of radius a quarter of a mile. The radian measure of the angle is $\frac{6}{1320} = \frac{1}{220}$. Taking 1 radian as 3438′ (see No. 3), the angle in minutes is $3438/220 \simeq 15{\cdot}6'$.]

17. A halfpenny whose diameter is 1 in. is held vertically; what angle does it subtend at a horizontal distance of (i) 6 ft., (ii) 22 yds.?

18. The sun's diameter subtends at the earth an angle of 32′. If the sun's distance is 93 million miles, what is its diameter?

19. The moon's diameter is 2160 miles and its average distance 240,000 miles. What angle does the moon's diameter subtend at the earth? What can you say about the *apparent* sizes of the sun and moon as seen from earth?

20. A ship, steaming due W., is observed to be 10,000 yds. due N. of a shore battery, from which it subtends an angle of 45 minutes. What is the approximate length of the ship?

21. For measuring the distance travelled, explorers sometimes set up a pole of known height at their last halt and measure the angle subtended at their next halt. If the pole is 6 ft. high and subtends 1′20″ at the next halt, what distance has been covered?

22. The diameter of Betelgeuse is 216 million miles, and its distance is 166 light years. Calculate the angle subtended at the earth by the diameter of this star. [A *light year* is the distance light travels in a year, and the speed of light is 186,000 miles per second.]

23. The *parallax* of a star is the angle subtended at the star by the radius of the earth's orbit (its almost circular path round the sun). The parallax of "Proxima Centauri," the nearest star, is 0·758 seconds. Find its approximate distance from the sun. [Radius of earth's orbit, 93 million miles.]

24. A range-finder consists essentially of two telescopes at the ends of an arm, one being fixed at right angles to the arm and the other movable. When a distant object is in view in both telescopes the angle between the two is measured. If the arm is 10 ft. long, what angles correspond to ranges of (i) 10,000 yds., (ii) 10 miles? (Fig. 119.)

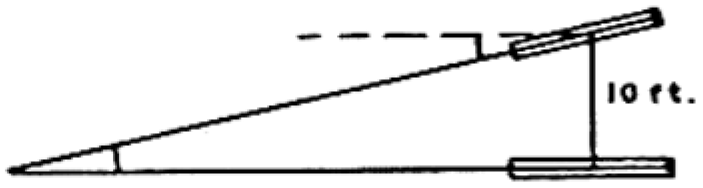

FIG. 119.

25. If θ is small, we may use θ as an approximation for $\sin\theta$ if θ is in radians. To find an approximation to $\cos\theta$, proceed as follows:—

(i) Using $\cos^2\theta + \sin^2\theta = 1$, express $\cos\theta$ in terms of $\sin\theta$.

(ii) Show, by comparing their squares, that

$$\sqrt{1-\theta^2} \simeq 1 - \tfrac{1}{2}\theta^2$$

if θ^4 is ignored.

[Note that if θ is small, *e.g.*, ·01, θ^4 will be very small, ·00000001.]

(iii) Deduce that $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$.

A beginning has been made with two important series which are learnt later:—

$$\sin\theta = \theta - \frac{\theta^3}{1\,.\,2\,.\,3} + \frac{\theta^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \ldots,$$

$$\cos\theta = 1 - \frac{\theta^2}{1\,.\,2} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} - \ldots$$

From these series the sine and cosine of even large angles can be calculated.

Large Angles

So far we have defined the trigonometrical ratios only for angles between 0° and 180°. This is all that is needed for angles of a triangle, but angles up to 360° are in use as bearings (p. 34) and these and larger angles may be used to express amount of rotation,* while negative angles will express rotation the opposite way.

What can we mean if we speak of the sine or cosine of such angles?

In defining the sine and cosine of angles greater than 180° we use the coordinates of the end of the radius whose turning traces out the angle, as explained on p. 36 for angles up to 180°.

* *E.g.*, through what angles have the minute and hour hands of a clock turned between midnight and 4.20 p.m.?

In Fig. 120, if r is the radius of the circle and (x,y) the coordinates of P,

$$\sin\theta = y/r \qquad \cos\theta = x/r.$$

Since r is not given any sign, the signs of $\sin\theta$ and $\cos\theta$ are those of y and x respectively.

For the angle shown in the figure the sine and cosine are both negative since y and x are negative. But if OP rotated further, into the fourth quadrant, the sine would still be negative but the cosine would become positive.

Negative Angles

In Fig. 121 angle XOP is formed by an anticlockwise rotation from OX to OP, and is considered as a *positive* angle θ. The angle XON is formed by a clockwise rotation from OX to ON, and is considered a *negative* angle θ.

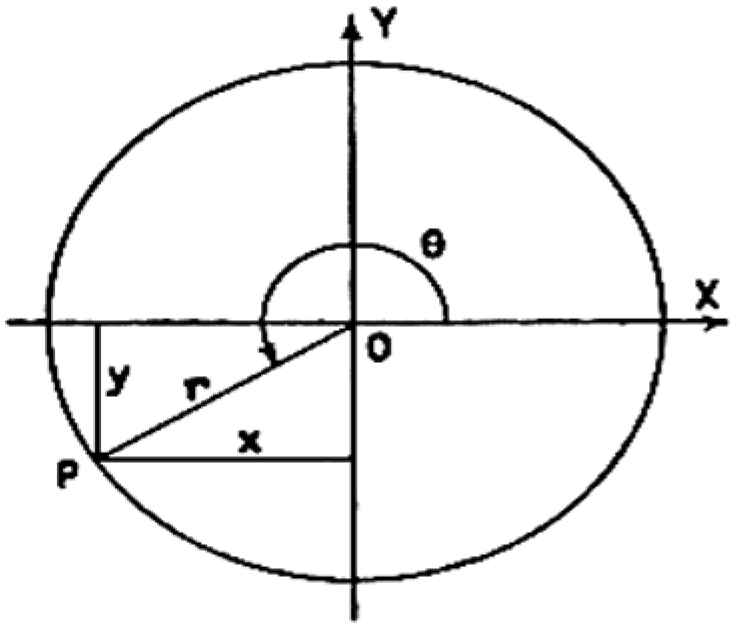

FIG. 120.

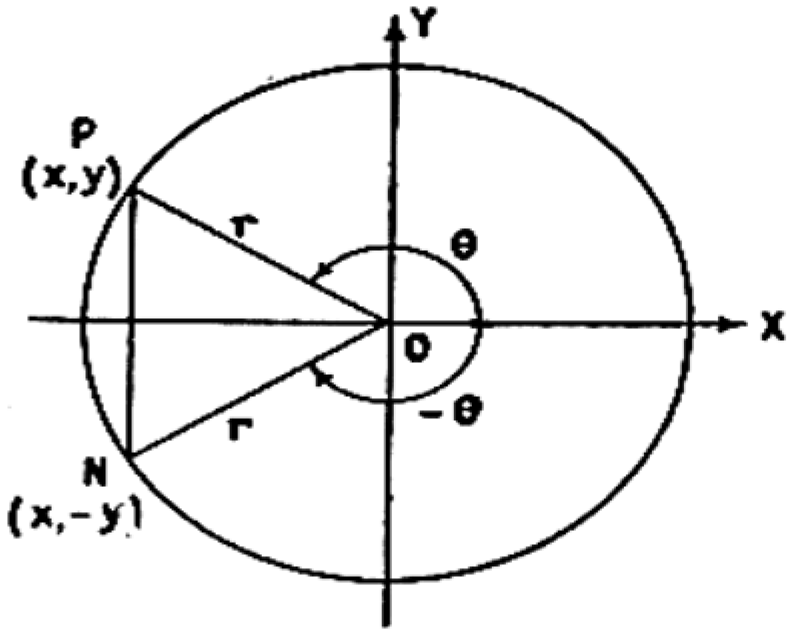

FIG. 121.

There are simple relations between the coordinates of P and N, for whatever the size of θ the points P and N are always in the same vertical line. Hence if the coordinates of P are (x,y), those of N are $(x,-y)$.

$$\therefore\ \sin(-\theta) = -y/r = -\sin\theta$$
$$\cos(-\theta) = x/r = \cos\theta.$$

Stretching the Trigonometrical Tables

Our tables cover only the range 0° to 90°, and we found the sines and cosines of obtuse angles by means of the relations $\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$. We need formulæ to help us with angles from 180° to 360°.

To add 180° to an angle XOP is to give OP a further rotation of 180°, in an anticlockwise direction, to the position OQ. Then, in Fig. 122, POQ is a diameter, and if P is (x, y) Q is clearly $(-x, -y)$. Since $\angle XOQ = 180° + \theta$, we get

$$\sin(180° + \theta) = -y/r = -\sin\theta$$
$$\cos(180° + \theta) = -x/r = -\cos\theta.$$

These formulæ are true whatever the size or sign of θ. If 180° is added to an angle of $-\theta$, we get

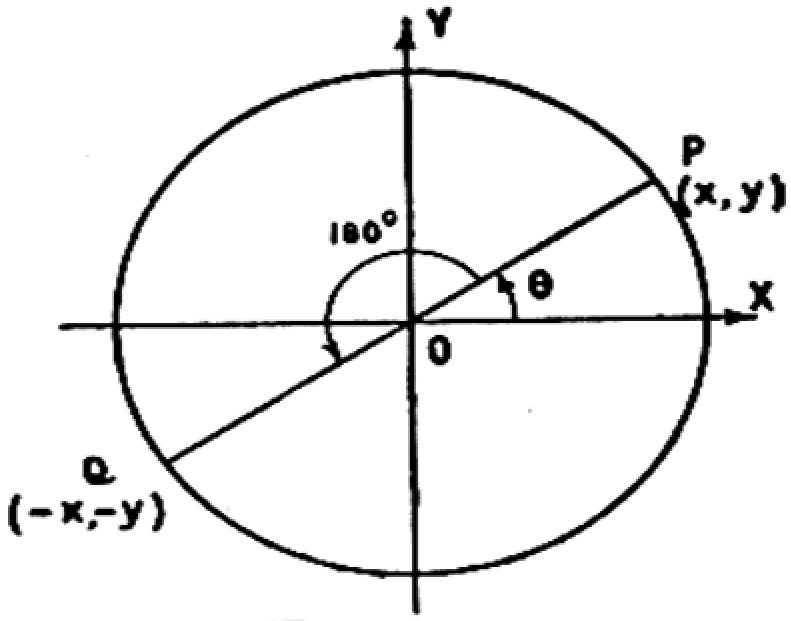

FIG. 122.

$$\sin(180° - \theta) = -\sin(-\theta) = \sin\theta$$
$$\cos(180° - \theta) = -\cos(-\theta) = -\cos\theta.$$

Example : To find sin 250° and cos 310°

$$\sin 250° = \sin(180° + 70°) = -\sin\ 70° = -{\cdot}9397$$
$$\cos 310° = \cos(180° + 130°) = -\cos 130° = +\cos 50° = {\cdot}6428.$$

The addition or subtraction of 360° *makes no difference to the sine, cosine or tangent,* for it brings the revolving radius back to the same place again.

The Complete Sine Curve

Since $\sin\theta = y/r = MP/OP$ in Fig. 123, and OP is constant, the graph of $\sin\theta$ is easily drawn by marking the height MP for different angles. If the radius OP is taken as the unit then $\sin\theta$ is equal to MP. The part of the graph corresponding to angles from 180° to 360° is clearly the same as the part for 0° to 180° except that it is upside down, thus illustrating the relation $\sin(180° + \theta) = -\sin\theta$.

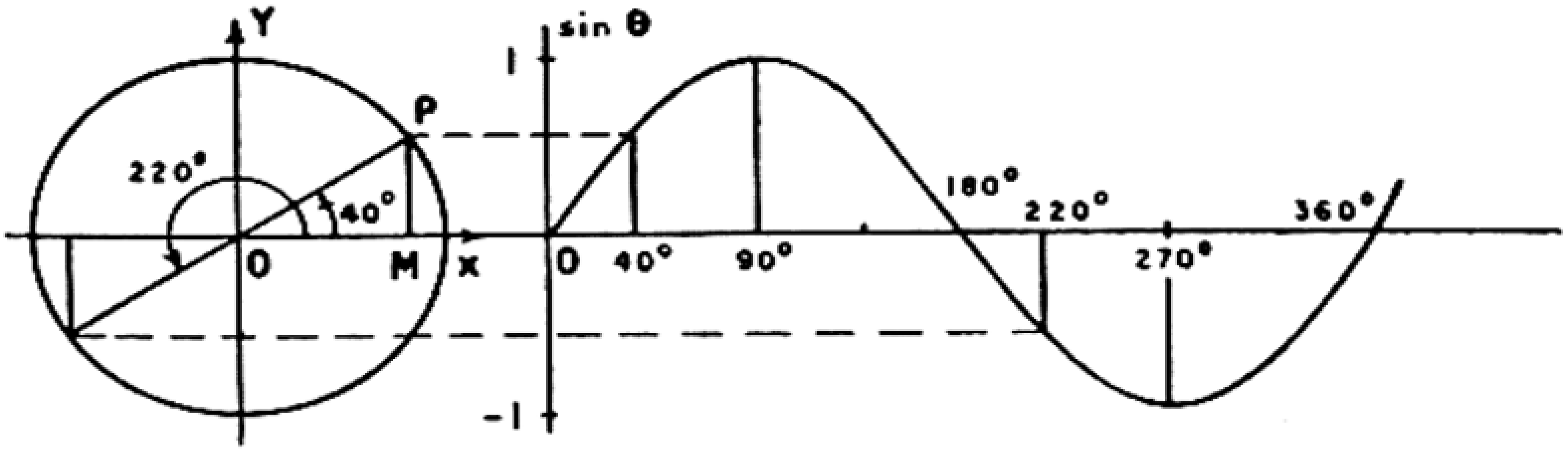

FIG. 123.

Beyond 360° the curve begins to repeat the pattern ; it is obvious that $\sin(360° + \theta) = \sin\theta$.

The Complete Cosine Curve

In Fig. 123 the "step to the right," OM, represents the cosine if the circle has unit radius. This is awkwardly placed for the actual construction of the cosine graph, so we imagine OP rotated through 90° to the position OQ, and $QN \perp OY$. Then $\triangle NOQ \equiv \triangle MOP$ and ON = OM, so that $ON = \cos\theta$ and the graph can be drawn by marking the height ON for different values of θ (Fig. 124).

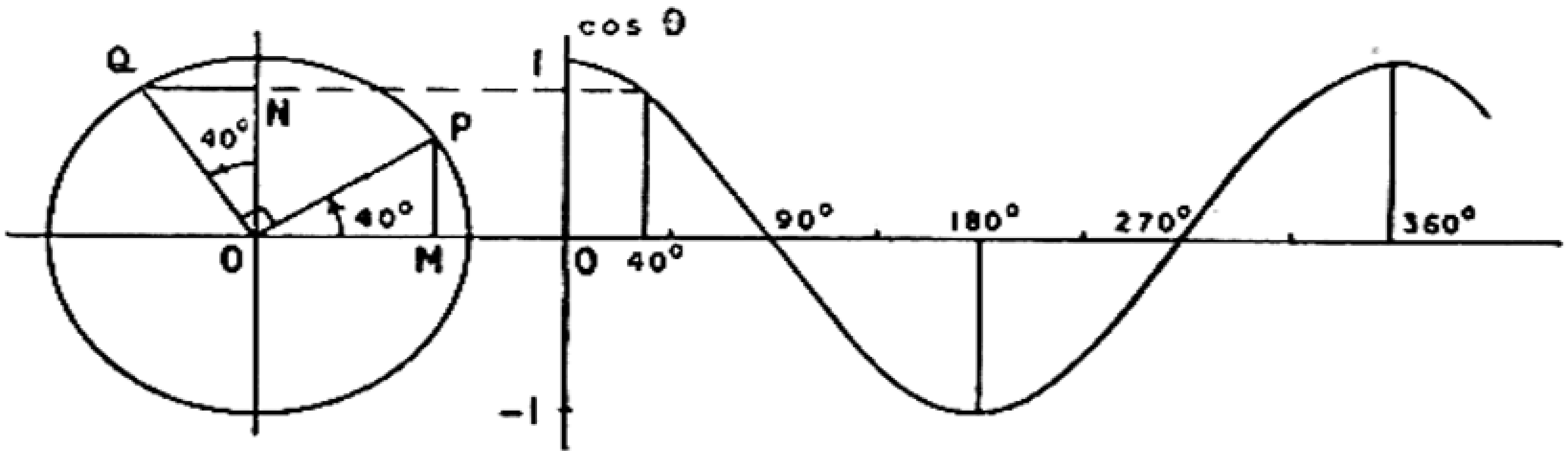

FIG. 124.

The cosine begins at the value reached by the sine at 90°, so that the curve begins at the "crest of the wave." This suggests the relation $\sin(90° + \theta) = \cos\theta$, which will be proved later.

As in the case of the sine, the curve begins to repeat the pattern after 360°, and the part from 180° to 360° is the same as the part from 0° to 180° but upside down, illustrating the relations

$$\cos(360° + \theta) = \cos\theta, \quad \cos(180° + \theta) = -\cos\theta.$$

Examples—54

1. Through what angle does the minute hand of Big Ben turn in 24 hours?

2. Using the coordinate definitions, and the relation $\tan\theta = \sin\theta/\cos\theta$, copy and complete the following table :—

	1st quadrant 0° to 90°	2nd quadrant 90° to 180°	3rd quadrant 180° to 270°	4th quadrant 270° to 360°
sine	+	+		
cosine	+	−		
tangent	+	−		

3. The diagram, Fig. 125, is often used to help fix the sign of a ratio. The letters * stand for All, Sine, Tangent, Cosine, and indicate which ratios are positive in each quadrant. Check this from the table of No. 2.

4. Using the formulæ for the sines and cosines of these angles, obtain similar formulæ for tan $(-\theta)$, tan $(180^\circ + \theta)$, tan $(180^\circ - \theta)$, tan $(360^\circ + \theta)$, tan $(360^\circ - \theta)$.

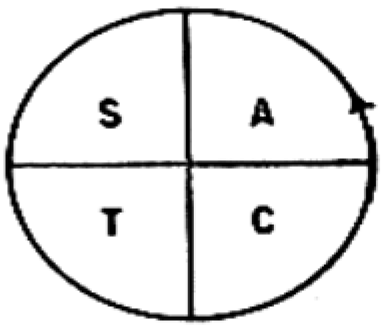

FIG. 125.

5. Find the values of the following, using the tables :—

(i) sin 200°, (ii) cos 220°, (iii) tan 240°, (iv) sin 260°, (v) cos 280°, (vi) tan 300°, (vii) sin 320°, (viii) cos 340°, (ix) sin 380°, (x) tan 160°.

6. Using the sine graph, Fig. 123, answer the following :—

(i) What angle has the same sine as 40° ?
(ii) What angle greater than 360° has the same sine as 40°?
(iii) What is the next angle whose sine is the same ?
(iv) Write down two more angles whose sine is the same as sin 40°.
(v) Write down three angles with the same sine as 220°.

7. Use the cosine graph, Fig. 124, to answer the following :

(i) What other angle has the same cosine as 40° ?
(ii) Give two larger angles with the same cosine as 40°.
(iii) Give two angles whose cosines are $+\frac{1}{2}$.
(iv) ,, ,, ,, ,, ,, ,, $-\frac{1}{2}$.
(v) Sketch the graph of cos $(90^\circ + \theta)$.

8. Sketch the graphs of (*a*) sin θ, (*b*) cos θ for values of θ from -180° to $+180^\circ$.

9. Make a table of the values of tan θ for the following angles : 0°, 30°, 45°, 60°, [90°], 120°, 135°, 150°, 180°, 210°, 225°, 240°, [270°], 300°, 315°, 330°, 360°, and draw a graph, taking 1 in. for 60° and 1 in. for one unit of tan θ. The values of tan 90° and tan 270° are indeterminate, but the trend of the curves can be seen.

10. The sine and cosine graphs repeat their "pattern" every 360°, and so the sine and cosine are said to have a "period" of 360°. What is the period of tan θ ? Use the graph of No. 9.

---

* Starting with C they spell CAST.

11. Copy and complete the following table, and use it to draw the graph of $(\sin\theta + \cos\theta)$:—

θ	0°	30°	and at 30° intervals	330°	360°
$\sin\theta$	0	·5	up to	$-$ ·5	0
$\cos\theta$	1	·87		$+$ ·87	1
$\sin\theta + \cos\theta$	1	1·37		$+$ ·37	1

12. *Additon of 90° to an angle:*

In Fig. 126 the lines P_1OP_3, P_2OP_4 are at right angles, and the angles P_1M_1O, etc., are right angles. The triangles P_1M_1O, etc., will then be congruent.

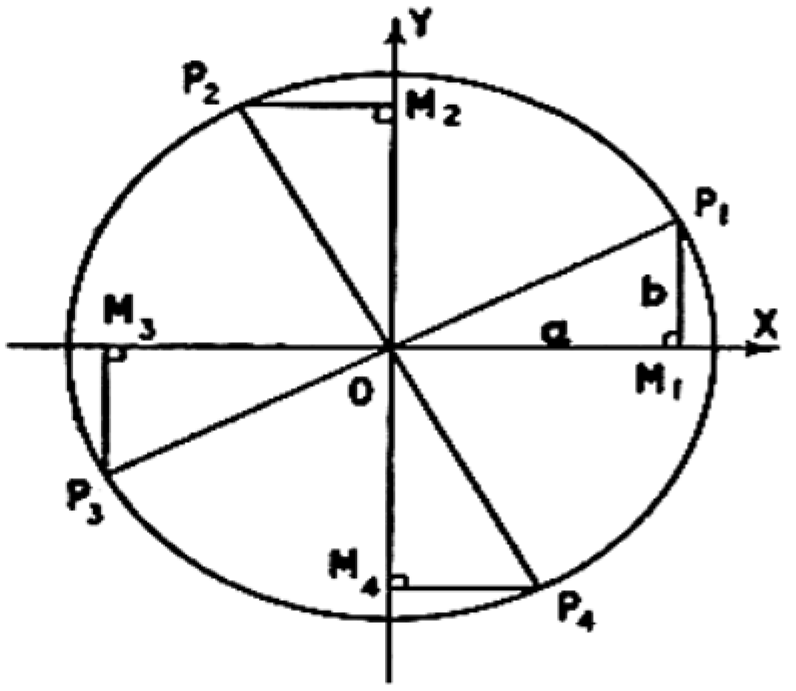

FIG. 126.

(i) If the coordinates of P_1 are (a, b), what are those of P_2, P_3, P_4 ?

(ii) If P is any one of the points P_1, P_2, P_3, P_4, and N is the next one (moving anticlockwise), show that $x_N = -y_P$ and $y_N = x_P$ in all four cases. (x_N means the x-coordinate of N, etc.).

(iii) If $\angle XOP = \theta$, then $\angle XON = 90° + \theta$. Prove that $\sin(90° + \theta) = \cos\theta$, $\cos(90° + \theta) = -\sin\theta$.

(iv) By taking $\angle XOP = -\theta$, which makes $\angle XON = 90° + (-\theta)$, deduce

$$\sin(90° - \theta) = \cos\theta, \qquad \cos(90° - \theta) = \sin\theta.$$

13. Deduce, from Ex. 12, formulæ for $\tan(90° + \theta)$ and $\tan(90° - \theta)$.

14. By writing $270° + \theta$ as $180° + (90° + \theta)$ deduce formulæ for $\sin(270° + \theta)$, $\cos(270° + \theta)$, $\sin(270° - \theta)$, $\cos(270° - \theta)$.

[*Disregarding signs*, the results of Nos. 12, 13 and 14 are included in the rule " If the sum or difference of two angles is 90° or 270°, the sine of either angle is equal to the cosine of the other." Instead of sine and cosine the ratios tangent and cotangent, or secant and cosecant, can be used.]

15. In the following statements the sign and the (acute) angle are missing; complete them :—

(i) $\sin 200^\circ = \ldots \cos \ldots$ (ii) $\sin 300^\circ = \ldots \cos \ldots$
(iii) $\tan 150^\circ = \ldots \cot \ldots$ (iv) $\sec 140^\circ = \ldots \operatorname{cosec} \ldots$
(v) $\cos 200^\circ = \ldots \sin \ldots$ (vi) $\cos 300^\circ = \ldots \sin \ldots$

[Sol'n. of (i) $\sin 200^\circ$ is negative, from a diagram or Fig. 125. The difference between 200° and 270° is 70°, so

$$\sin 200^\circ = -\cos 70^\circ.]$$

16. In Fig. 120 what is the relation between x, y and r? Hence prove that the following relations are true for any value of θ :—

$$\cos^2\theta+\sin^2\theta=1 \qquad \sec^2\theta=1+\tan^2\theta \qquad \operatorname{cosec}^2\theta=1+\cot^2\theta.$$

[Hint : $\sec\theta = r/x$;

$$1+\tan^2\theta = 1 + y^2/x^2 = (x^2+y^2)/x^2.]$$

17. Give the values of (i) $\sin \pi$, (ii) $\cos 3\pi/2$, (iii) $\sin 4\pi/3$, (iv) $\cos 2\pi$, (v) $\tan 5\pi/4$, (vi) $\sin 5\pi/3$, (vii) $\sec \pi$, (viii) $\operatorname{cosec} 11\pi/6$.

Summary of General Formulæ

$$\begin{array}{ll}
\sin(-\theta) = -\sin\theta & \cos(-\theta) = \cos\theta \\
\sin(180^\circ-\theta) = \sin\theta & \cos(180^\circ-\theta) = -\cos\theta \\
\sin(180^\circ+\theta) = -\sin\theta & \cos(180^\circ+\theta) = -\cos\theta \\
\sin(360^\circ-\theta) = -\sin\theta & \cos(360^\circ-\theta) = \cos\theta \\
\sin(90^\circ-\theta) = \cos\theta & \cos(90^\circ-\theta) = \sin\theta \\
\sin(90^\circ+\theta) = \cos\theta & \cos(90^\circ+\theta) = -\sin\theta
\end{array}$$

These may be quoted as "the $(-\theta)$ formula," etc.

The corresponding formulæ for the tangent are got from $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$.

Note that, if the *signs* of the right-hand sides are obtained from rough figures, these results are included in the rule : "If the sum or difference of two angles is 180° or 360° the sine (or cosine) of one is numerically equal to the sine (or cosine) of the other. If the sum or difference of two angles is 90° or 270° the sine (or cosine) of one is numerically equal to the *cosine* (*or sine*) of the other." Similarly for the other pairs, secant and cosecant, tangent and cotangent.

Period, Amplitude, Phase

The sine curve repeats its pattern every 360°. This is called its *period.* If a graph is drawn showing several complete periods it is recognised as the picture of a wave or a vibration. This particular form of vibration is that which is most common in nature. Those who study sound waves, the rise and fall of the tides, electricity, radio have much to do with sine curves. The period for $\sin 2x$ is 180°, for $\sin 3x$ is 120°.

Sine curves may differ in *amplitude.* For $a \sin x$ the amplitude is a; thus the graph of $2 \sin x$ has an amplitude double that of $\sin x$.

Sine curves also differ in *phase.* The graph of $\sin (x + \alpha)$ can be obtained by moving the graph of $\sin x$ a distance α to the left; α is said to be the difference in *phase* between the two curves. It is easily seen that the graph of $\cos x$ differs only in phase from that of $\sin x$ being the graph of $\sin (x + 90°.)$

Examples—55

1. Describe, using the phrases explained above, how the graphs of each of the following differs from that of $\sin x$:—
(i) $2 \sin (x + 30°)$. (ii) $\sin 2 (x + 20°)$. (iii) $3 \cos x$.
(iv) $a \sin (\omega x + \alpha)$.

2. A vibration is given by $y = a \sin k(t - t_0)$ where t represents time. What are its period and amplitude, and what is its phase difference from $y = a \sin kt$?

3. It is often convenient to work in *radians.* A *rough* approximation convenient for squared paper is to take an inch for the y unit and $1\frac{1}{2}$ in. for $x = 90°$; this corresponds to taking $\pi = 3$. Draw rough sketches of (i) $y = \sin x$ and $y = \sin 2x$ from $x = 0$ to $x = 2\pi$, (ii) $y = \sin 2x$ and $x = \sin 2y$ from $x = 0$ to $x = \pi$ and $y = 0$ to $y = \pi$.

4. If x is measured in radians, *sketch* the graph of $y = \sin x$ from $x = 0$ to $x = 10$. Where does the graph cut the x-axis? The wavelength of a wave is the distance from one crest to the next. What are the wavelengths of the following:

(i) $y = \sin x$, (ii) $y = \sin \pi x$,

(iii) $y = \sin \dfrac{\pi x}{c}$, (iv) $y = \sin \dfrac{2\pi x}{l}$?

5. In a simple dynamo the graph of current against time is a sine curve. If the current rises to a maximum of 2 amperes and is zero at regular intervals of $\frac{1}{100}$ second, find (i) the wavelength of the graph (in seconds), (ii) a formula of the type $C = A \sin nt$ which will fit the graph, C being the current in amperes and t the time in seconds.

Trigonometrical Equations

In solving trigonometrical equations it is usual to have to find one of the ratios, say $\sin \theta$, first and then to find the possible values of θ.

In the first of these processes there is considerable variety in the methods, and some resourcefulness is needed to select the right one, but the second process is straightforward and it is well to master it first. (Ex. 1 to 4.) Note that in the figure S,T,C mark the other points when the turning radius can stop to make an angle with the same sine, tangent, cosine respectively as the acute angle XOP (Fig. 127).

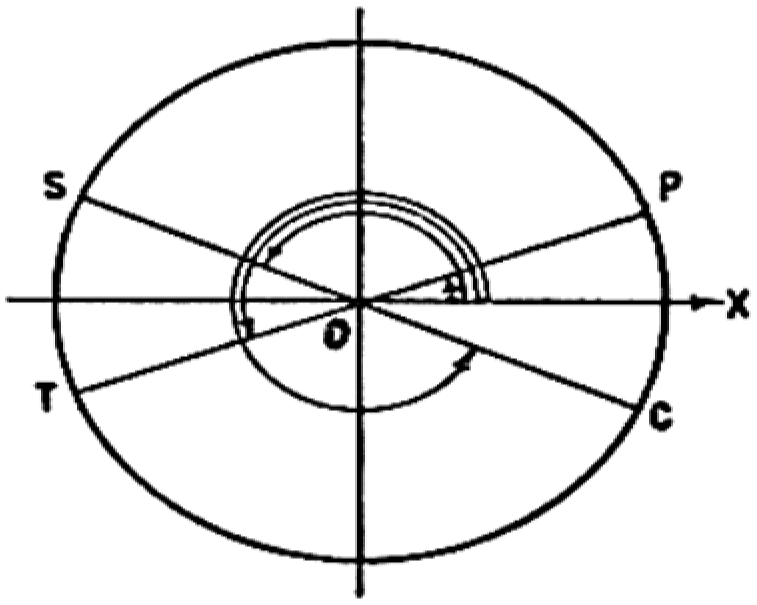

FIG. 127.

Examples—56

1. Find all the angles between 0° and 360° which satisfy (i) the equation $\sin \theta = \cdot 4$. (ii) the equation $\sin \theta = -\cdot 4$.

[Sol'n. (i) By the tables $\sin 23°35' = \cdot 4$. If $\angle$AOP is 23°35', $\angle$AOS $= 180° - 23° 35' = 156°25'$. Thus the required angles are 23°35' and 156°25'.

(ii) The angles whose sines are $\pm \cdot 4$ are $180° \pm 23°35'$ and $360° \pm 23°35'$. But angles whose sines are negative are in the third and fourth quadrants. So the required angles are $180° + 23°25'$, *i.e.*, XOT and $360° - 23°35'$, *i.e.*, XOS $\therefore$ the required angles are 203°35' and 336°25'].

2. Find all the angles between 0° and 360° which satisfy (i) $\cos \theta = \cdot 6$, (ii) $\tan \theta = -\cdot 7$, (iii) $\sec \theta = 2\cdot 5$.

3. Solve the equations (i) $\sin^2 \theta = \cdot 16$. (ii) $\cos^2 \theta = \cdot 04$.

[Sol'n. (i) This gives $\sin \theta = \pm \cdot 4$ and leads to all the four solutions of the two equations in Ex. 1].

4. Solve the following equations giving answers between $0°$ and $360°$:

(i) $\cos\theta = -\cdot 5$, (ii) $\sin\theta = \cdot 766$, (iii) $\tan\theta = -1$,
(iv) $\tan\theta = \cdot 404$, (v) $\cos\theta = 0$, (vi) $\operatorname{cosec}\theta = 1\cdot 325$.

[*The following, Ex. 5, 6, 7, are examples of the first part of the process only. It may help to replace the ratio required by x.*]

5. Find $\sin\theta$ from (i) $\operatorname{cosec}\theta + 2 = 0$, (ii) $3\sin\theta = \operatorname{cosec}\theta$, (iii) $6\sin^2\theta - 5\sin\theta = 1$.

6. Find $\cos\theta$ from (1) $\sec\theta = 4\cos\theta$, (ii) $2\cos^2\theta - \cos\theta = 0$, (iii) $2\cos^2\theta - \cos\theta = 1$.

7. Find $\tan\theta$ from (i) $12\sin\theta = 5\cos\theta$, (ii) $\tan\theta - 4\cot\theta = 0$, (iii) $2\tan^2\theta - 9\tan\theta = 5$.

[*In all that follow, both parts of the process of solution are needed.*]

8. Solve the equations (i) $3\cos\theta = 2\sin\theta$, (ii) $2\tan^2\theta - \tan\theta = 0$, (iii) $\cos^3\theta - 3\cos\theta = 0$, (iv) $\sin\theta(2\cos\theta - 1) = 0$, (v) $\cos\theta + \sec\theta = 2\cdot 5$, (vi) $5\cos\theta + 3\cos^2\theta = 2$, giving in each case all the solutions between $0°$ and $360°$. Note that such solutions as $\cos\theta = 2$ are impossible and must be rejected.

[In some equations the angle first found is not an answer. This makes care necessary to get the answers which are between $0°$ and $360°$].

9. Find the values of θ between $0°$ and $360°$ which make $\tan 3\theta = 1\cdot 6$.

[Sol'n. θ lies between $0°$ and $360°$ if 3θ lies between $0°$ and $1080°$. The acute angle whose tangent is $1\cdot 6 \simeq 58°$ (or briefly, $\tan^{-1} 1\cdot 6 = 58°$). The tangent is positive in the first and third quadrants, so the values of 3θ are $58°$, $58° + 180°$, $58° + 360°$, and so on. ($58° + n.180°$ where n is an integer.) The values of θ are $19\frac{1}{3}°$, $19\frac{1}{3}° + 60°$, $19\frac{1}{3}° + 120°$, and so on. ($19\frac{1}{3}° + n.60°$). *I.e.*, $19\frac{1}{3}°$, $79\frac{1}{3}°$, $139\frac{1}{3}°$, $199\frac{1}{3}°$, $259\frac{1}{3}°$, $319\frac{1}{3}°$.]

10. Find values of θ between $0°$ and $180°$ to satisfy the following equations :—(i) $\sin 4\theta = \cdot 5$, (ii) $\cos 3\theta = -\cdot 5$, (iii) $\tan 3\theta = 1$, (iv) $\sin 5\theta = 1$.

11. Solve the equations, giving any answers between 0° and 360° :—

(i) $\sin(30° + \theta) = \cdot 891$. [Write x for $(30° + \theta)$.]
(ii) $\sin(60° + \theta) = \frac{1}{2}$. (iii) $\sin(70° + \theta) = \cdot 66$.

[Sol'n of (ii). $(60° + \theta)$ would be 30° or 150° or 390°, etc. The first value makes θ negative, so the next two must be taken, giving 90° and 330°.]

12. Solve the following equations giving in each case the numerically smallest *negative* answer : (i) $\cos(60° + \theta) = \cdot 969$, (ii) $\cos(75° + \theta) = -\cdot 682$, (iii) $\tan 2\theta = 1$.

[In many cases the use of one of the Pythagorean relations $\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta = 1 + \tan^2\theta$, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ is needed to express the given equation in terms of one ratio.]

13. Solve the equations (giving answers between 0° and 180°) :—

(i) $2\cos^2\theta = 3\sin\theta$, (ii) $\sin^2\theta = \cos\theta - 1$,
(iii) $8\cos\theta = 3\sin^2\theta$, (iv) $\sec^2\theta = 2\tan\theta$.

[Sol'n. (i) if $x = \sin\theta$, $2 - 2x^2 = 3x$ or $(2x-1)(x+2) = 0$. $x = -2$ must be rejected, etc.]

14. Solve the equations giving answers between 0° and 180° :—

(i) $\operatorname{cosec}^2\theta = \cot\theta$, (ii) $\operatorname{cosec}^2\theta = 2\cot\theta$, (iii) $\operatorname{cosec}^2\theta = 3\cot\theta$.

[Prove that (i) is impossible. The quadratic obtained has no solutions. In (iii) take $\sqrt{5} = 2 \cdot 236$ and get two values of $\cot\theta$.]

15. Give the smallest positive and negative answers (other than $\theta = 0$) to the equations (i) $\sin\theta = \cdot 4(1 + \sin^2\theta)$, (ii) $\sin\theta + \operatorname{cosec}\theta = 4 \cdot 25$, (iii) $\tan\theta = 3\sin\theta$.

Equations which seem difficult or impossible by algebraic methods can always be solved by drawing graphs.

16. Solve the equation $\cos\theta = 1 + \sin 2\theta$ by drawing the graphs of $\cos\theta$ and of $1 + \sin 2\theta$ from 0° to 360°. Take 1 in. for 30° and 1 in. for one vertical unit. Tabulate and plot values at 30° intervals. The three intersections give the solutions.

17. Solve the equation of Ex. 16 by drawing the graph of $\cos\theta - \sin 2\theta$ and finding the values of θ for which $\cos\theta - \sin 2\theta = 1$, (*i.e.*, $\cos\theta = 1 + \sin 2\theta$).

From the same graph read the solutions of $\cos\theta - \sin 2\theta = \frac{1}{2}$.

ADDITION FORMULÆ

The "Addition Formulæ" which will now be proved are

$$\begin{aligned}\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi\\ \sin(\theta-\phi) &= \sin\theta\cos\phi - \cos\theta\sin\phi\\ \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi\\ \cos(\theta-\phi) &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

The formulæ summarised on p. 119 are particular cases of the above formulæ in which one of the angles is 90°, 180° or 360°.

A single proof, valid for angles of every possible size, is difficult and will not be given here. The two proofs given are satisfactory for acute angles and can be extended to particular cases of larger angles if necessary.

The addition formulæ are useful in calculating trigonometrical tables, they are needed in Calculus in order to differentiate the sine and cosine, and they are of frequent occurrence throughout most of the further mathematics that you may do.

Proofs of the Formulæ

I. This proof uses the double cosine formula

$$c = a\cos B + b\cos A$$

and also the formula $c = 2R\sin C$, etc. It is valid if θ and ϕ are positive and $\theta + \phi < 180°$.

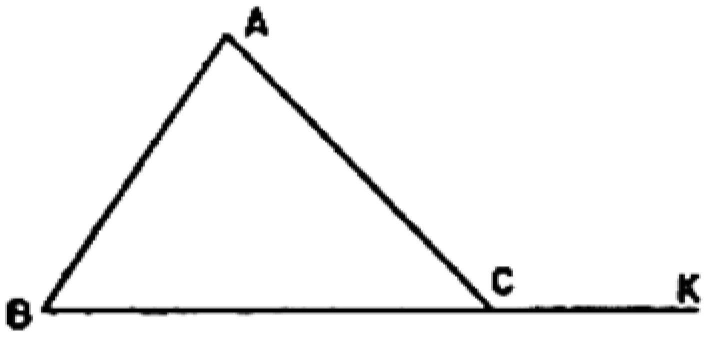

FIG. 128.

In Fig. 128, $\angle ACK = A + B$, and $\sin ACK = \sin ACB = \sin C$.

By the double cosine formula, $c = a\cos B + b\cos A$, and by the sine formula, $c = 2R\sin C$, etc.

$$\therefore\ 2R\sin C = 2R\sin A\cos B + 2R\sin B\cos A,$$

and, since $\sin(A+B) = \sin C$,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B.$$

(i) If θ and ϕ are positive acute angles, and $\theta > \phi$.

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

by putting $A = \theta$, $B = \phi$.

(ii) If we put $A = 180° - \theta$ and $B = \phi$, then $A + B$ is less than 180°.

$$\sin(A+B) = \sin(180° - \theta + \phi) = \sin(180° - \overline{\theta - \phi}) = \sin(\theta - \phi)$$

and $\sin A \cos B + \cos A \sin B$

$$= \sin(180^\circ - \theta)\cos\phi + \cos(180^\circ - \theta)\sin\phi$$

$$\therefore \sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

(iii) Now $\cos(\theta + \phi) = \sin(90^\circ + \theta + \phi)$. If we put $A = 90^\circ + \theta$, $B = \phi$, then $A + B$ is less than 180° if $\theta + \phi$ is less than 90°, and substituting

$$\sin(90^\circ + \theta + \phi) = \sin(90^\circ + \theta)\cos\phi + \cos(90^\circ + \theta)\sin\phi$$

$$\therefore \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

(iv) Similarly $\cos(\theta - \phi) = \sin(90^\circ + \theta - \phi)$. If $A = 90^\circ - \phi$, $B = \theta$ then $A + B$ is less than 180°, and, substituting,

$$\sin(90^\circ + \theta - \phi) = \sin(90^\circ - \phi)\cos\theta + \cos(90^\circ - \phi)\sin\theta$$

$$\therefore \cos(\theta - \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

II. This proof uses the area formula $\triangle = \frac{1}{2}ab\sin C$ and the cosine formula $a^2 = b^2 + c^2 - 2bc\cos A$.

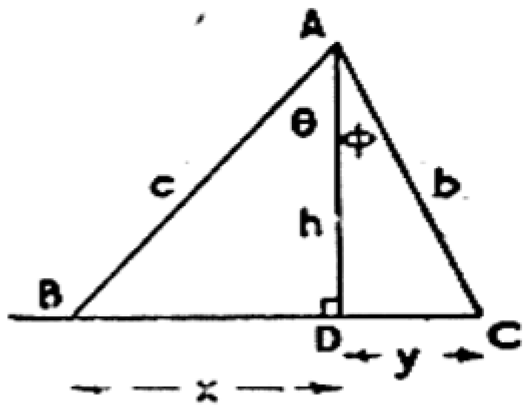

FIG. 129.

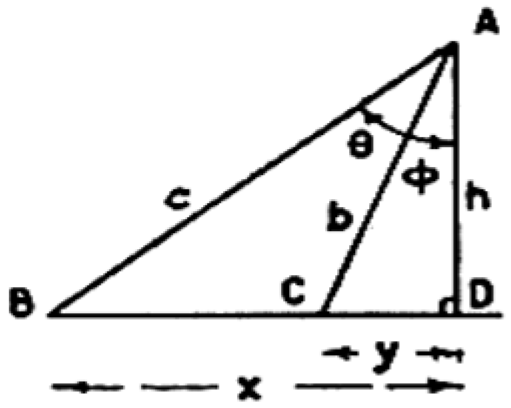

FIG. 130.

In the above figures AD is perpendicular to BC.

In the proofs the upper signs refer to Fig. 129, the lower to Fig. 130.

(i) $\angle BAC = \theta \pm \phi$, $\triangle ABC = \triangle ABD \pm \triangle ACD$

$$\therefore \tfrac{1}{2}bc\sin(\theta \pm \phi) = \tfrac{1}{2}ch\sin\theta \pm \tfrac{1}{2}bh\sin\phi.$$

Multiplying both sides by 2 and dividing by bc,

$$\sin(\theta \pm \phi) = \frac{h}{b}\sin\theta \pm \frac{h}{c}\sin\phi$$

$$= \cos\phi\sin\theta \pm \cos\theta\sin\phi.$$

(ii) Also,

$$2bc\cos A = 2bc\cos(\theta \pm \phi) = b^2 + c^2 - (x \pm y)^2$$

$$= h^2 + y^2 + h^2 + x^2 - x^2 \mp 2xy - y^2 = 2h^2 \mp 2xy$$

$$\therefore \cos(\theta \pm \phi) = \frac{h^2}{bc} \mp \frac{xy}{bc} = \frac{h}{c}\cdot\frac{h}{b} \mp \frac{x}{c}\cdot\frac{y}{b}$$

$$= \cos\theta\cos\phi \mp \sin\theta\sin\phi.$$

Examples—57

1. Show that the formulæ for $\cos(\theta - \phi)$ and $\cos(\theta + \phi)$ can be proved by means of II (i) if $\angle BAD$ is taken as $(90^\circ - \theta)$, *i.e.*, if $90^\circ - \theta$ is written instead of θ in the formulæ for $\sin(\theta + \phi)$ and $\sin(\theta - \phi)$.

2. The two terms on the right-hand side of each of the addition formulæ can be shown as actual lengths in the proofs suggested by Figs. 131 and 132.

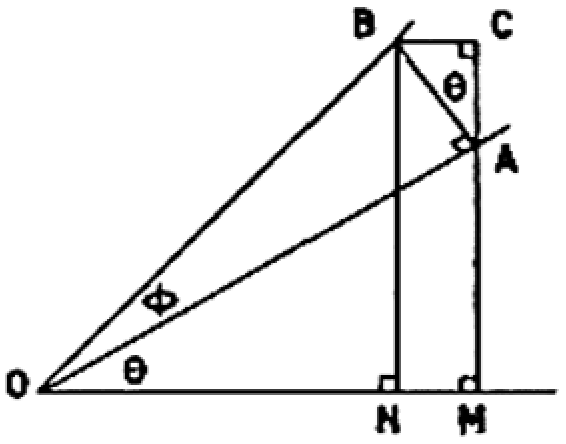

FIG. 131.

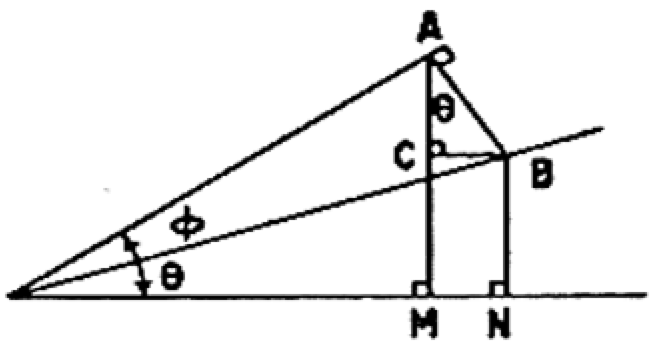

FIG. 132.

Taking OB as unit length, express in terms of θ and ϕ the lengths of AB, BC, NM, CA, OA, AM, OM, showing them on a larger copy of each figure. Find lengths on the two figures which are equal to $\sin(\theta+\phi)$, $\cos(\theta+\phi)$. $\sin(\theta-\phi)$, $\cos(\theta-\phi)$ and complete the proofs.

3. Show that formula for $\sin(\theta+\phi)$ holds if θ and ϕ are obtuse, as follows: suppose $\theta = 180° - \alpha$, $\phi = 180° - \beta$; then α and β are both acute.

$$\sin(\theta+\phi) = \sin(360° - \alpha - \beta) = \sin(360° - \overline{\alpha+\beta})$$
$$= -\sin(\alpha+\beta).$$

Show that $\sin\theta\cos\phi + \cos\theta\sin\phi$
$$= -\sin\alpha\cos\beta - \cos\alpha\sin\beta = -\sin(\alpha+\beta).$$

4. Writing

$$\tan(\theta+\phi) = \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi},$$

and dividing above and below by $\cos\theta\cos\phi$, obtain the formula for $\tan(\theta+\phi)$. Derive the formula for $\tan(\theta-\phi)$ by a similar method.

Verifications

The values of the sine, cosine and tangent for special angles like 0°, 30°, 45°, 60° and 90° will be needed for this. The diagrams, Fig. 133, will help. Examples 5 to 12 afford good practice in dealing with $\sqrt{2}$ and $\sqrt{3}$.

5. Taking $\theta = 60°$, $\phi = 30°$, verify the formulæ for $\sin(\theta \pm \phi)$, $\cos(\theta \pm \phi)$.

[Sol'n. $\sin(\theta+\phi) = \sin 90° = 1$;

$$\sin\theta\cos\phi + \cos\theta\sin\phi = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{3}{4} + \tfrac{1}{4} = 1].$$

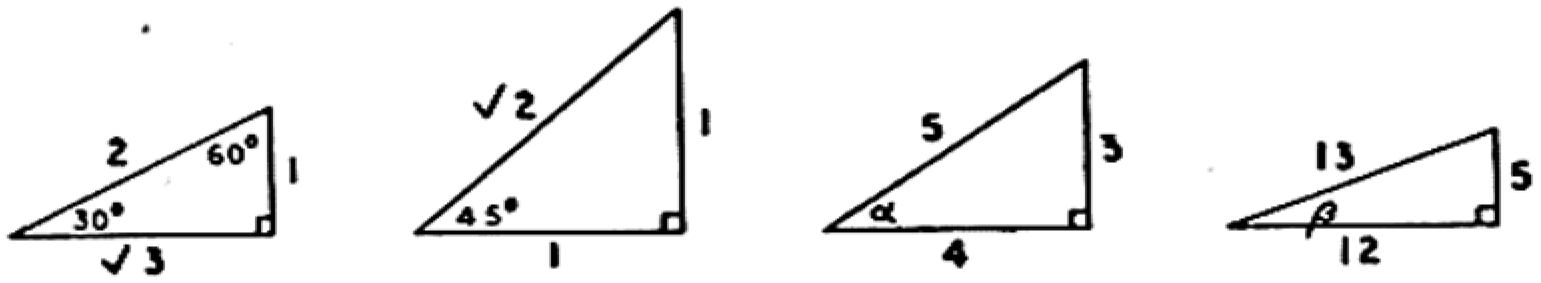

FIG. 133.

6. Verify the same formulæ when (i) $\theta = 45° = \phi$. (ii) $\theta = 90°$, $\phi = 0$.

7. Show that $\sin 15° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ in two ways, (i) taking $\theta = 60°$, $\phi = 45°$, (ii) taking $\theta = 45°$, $\phi = 30°$ Compare with the tables.

[Hint for last part :—

$$\frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\cdot 3660}{\sqrt{2}} = \frac{\cdot 366 \times \sqrt{2}}{2} = \cdot 183 \times 1 \cdot 414.]$$

8. Calculate $\cos 15°$ in the same ways as in Ex. 7.

9. Show that $\sin 75° = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$.

10. Test the tangent formula for $\tan(\theta + \phi)$ by taking $\theta = \phi = 30°$, and that for $\tan(\theta - \phi)$ by taking $\theta° = 60°$, $\phi = 30°$.

$$\left[\text{Use} \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) \Big/ \left(1 + \sqrt{3} \,.\, \frac{1}{\sqrt{3}}\right) = \frac{3 - 1}{\sqrt{3}} \Big/ 2 = \frac{1}{\sqrt{3}}\right].$$

12. Show that $\tan 75° = 2 + \sqrt{3}$ and $\tan 15° = 2 - \sqrt{3}$, taking $\theta = 45°$, $\phi = 30°$.

$$\left[\left(1 + \frac{1}{\sqrt{3}}\right) \Big/ \left(1 - \frac{1}{\sqrt{3}}\right) = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}\right.$$
$$\left. = \frac{(\sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{3 + 2\sqrt{3} + 1}{3 - 1} = 2 + \sqrt{3}\right].$$

Substitutions

Use Fig. 133 in Nos. 13, 14, 15.

13. Evaluate $\sin(\alpha + \beta)$, $\sin(\alpha - \beta)$, $\cos(\alpha + \beta)$, $\cos(\alpha - \beta)$, $\tan(\alpha + \beta)$, $\tan(\alpha - \beta)$.

14. Show that $5 \sin(\theta + \alpha) = 4 \sin\theta + 3 \cos\theta$, and express $5 \cos(\theta - \alpha)$ in the same form.

15. Evaluate $\sin(45° + \alpha)$ and $\cos(45° + \beta)$.

16. Show that $\sin(45^\circ + \theta) = (\sin\theta + \cos\theta)/\sqrt{2} = \cos(45^\circ - \theta)$. Why are $(\sin 45^\circ + \theta)$ and $\cos(45^\circ - \theta)$ equal?

17. Show that (i) $\cos(45^\circ + \theta) + \cos(45^\circ - \theta) = \sqrt{2}\cos\theta$.

(ii) $\cos(60^\circ + \theta) + \cos(60^\circ - \theta) = \cos\theta$.

18. Show that $\tan(45^\circ + \theta) = (1 + \tan\theta)/(1 - \tan\theta)$ and find a similar expression for $\tan(45^\circ - \theta)$.

Double and Triple Angles

19. Obtain the following results from the addition formulæ by making $\phi = \theta$:—

$$\mathbf{\sin 2\theta = 2\sin\theta\cos\theta} \qquad \mathbf{\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}}.$$

$$\mathbf{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

What result is obtained by making $\phi = \theta$ in the formula for $\cos(\theta - \phi)$?

20. Show that the formula for $\cos 2\theta$ can be written as

$$\cos 2\theta = 2\cos^2\theta - 1$$

or as

$$\cos 2\theta = 1 - 2\sin^2\theta.$$

21. Test the formulæ for $\sin 2\theta$ and $\cos 2\theta$ when $\theta = 30^\circ$ and 45° and calculate $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$, $\sin 2\beta$, $\cos 2\beta$, $\tan 2\beta$ with the notation of Fig. 133.

22. (i) Prove that $\mathbf{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$ by taking $\phi = 2\theta$.

[$\sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$; substitute for $\cos 2\theta$ and $\sin 2\theta$].

(ii) Express $\cos 3\theta$ in terms of $\cos\theta$, and test the formula by taking $\theta = 30^\circ$.

23. Use the results of Ex. 22 to find $\sin 3\alpha$ and $\cos 3\alpha$ when α is the acute angle whose sine is 0·6.

24. Prove that $\tan 3\theta = (3\tan\theta - \tan^3\theta)/(1 - 3\tan^2\theta)$. [Express $\tan 3\theta$ as $\tan(\theta + 2\theta)$. The work is simplified by using x for $\tan\theta$.]

Take (i) $\tan\theta = \cdot 5$, (ii) $\tan\theta = \cdot 8$ and in each case calculate $\tan 3\theta$.

25. Use the formula $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 72^\circ = \cdot 3090$ to calculate (i) $\cos 144^\circ$ as $2\cos^2 72^\circ - 1$, (ii) $\cos 36^\circ$ from $2\cos^2 36 = 1 + \cos 72^\circ$.

How do your results check each other?

26. Find $\sin 35^\circ$ from $\cos 2\theta = 1 - 2\sin^2\theta$ and $\cos 70^\circ = \cdot 3420$.

27. Show that the double angle formula used in Ex. 25, 26 may be changed into

$$\mathbf{\sin^2 \tfrac{1}{2}\theta = \tfrac{1}{2}(1 - \cos \theta)} \text{ and } \mathbf{\cos^2 \tfrac{1}{2}\theta = \tfrac{1}{2}(1 + \cos \theta)}$$

by writing $\frac{1}{2}\theta$ for θ.

Check these results by using $\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta = 1$, $\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta = \cos \theta$.

Deduce a formula for $\tan^2 \frac{1}{2}\theta$.

28. Verify the formulæ of Ex. 27 by putting $\frac{1}{2}\theta = 30°$.

Also verify it by putting $\frac{1}{2}\theta = 15°$ and using the results of Ex. 7 and Ex. 8.

29. Use the formulæ of Ex. 27 with $\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$ to show that

$$\frac{1 - \cos \theta}{\sin \theta} = \tan \tfrac{1}{2}\theta \qquad \frac{\sin \theta}{1 + \cos \theta} = \tan \tfrac{1}{2}\theta.$$

30. Prove that $(\sin \frac{1}{2}\theta + \cos \frac{1}{2}\theta)^2 = 1 + \sin \theta$,

$$(\sin \tfrac{1}{2}\theta - \cos \tfrac{1}{2}\theta)^2 = 1 - \sin \theta.$$

[If the square roots of both sides of these equations are taken, care is needed as to sign.]

If $\theta = 280°$ why must $\sin \frac{1}{2}\theta + \cos \frac{1}{2}\theta = -\sqrt{(1 + \sin \theta)}$ while $\sin \frac{1}{2}\theta - \cos \frac{1}{2}\theta = +\sqrt{(1 - \sin \theta)}$?

The t-Formulæ

If $t = \tan \frac{1}{2}\theta$ it is possible to express $\sin \theta$, $\cos \theta$ and $\tan \theta$ in terms of t by means of comparatively simple formulæ *without square roots*. As a result several trigonometrical problems can be replaced by algebraical ones.

We know that $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$, so that $\tan \theta = \dfrac{2t}{1 - t^2}$.

In Fig. 134, $\angle ABC$ is a right angle.

$\angle BAC = \theta$ if $BC = 2t$
and $AB = 1 - t^2$.

Then, by Pythagoras' theorem,

$$\begin{aligned} AC^2 &= (1 - t^2)^2 + (2t)^2 \\ &= 1 - 2t^2 + t^4 + 4t^2 \\ &= 1 + 2t^2 + t^4. \end{aligned}$$

$\therefore\ AC = 1 + t^2$.

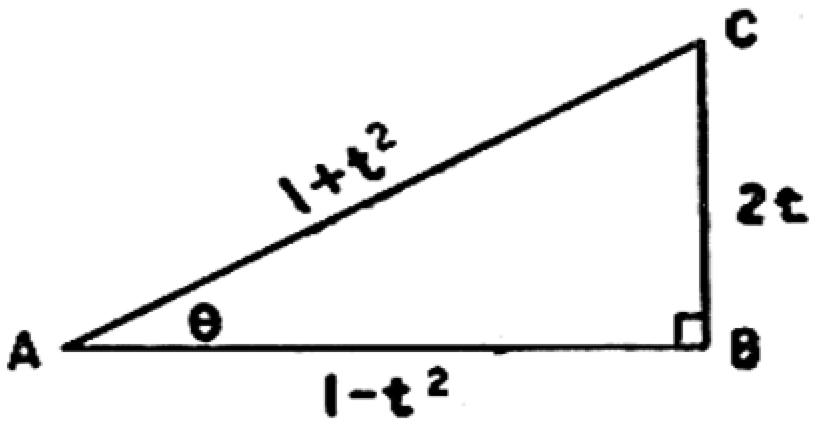

FIG. 134.

$$\therefore\ \mathbf{\sin \theta = \frac{2t}{1 + t^2}} \qquad \mathbf{\cos \theta = \frac{1 - t^2}{1 + t^2}}.$$

31. Express in terms of t (i) cosec θ, (ii) sec θ, (iii) cot θ, (iv) cot $\frac{1}{2}\theta$, (v) tan $(45^\circ + \frac{1}{2}\theta)$, (vi) sec θ + tan θ, (vii) cosec θ + cot θ, (viii) cosec θ — cot θ, (ix) $\sec^2 \frac{1}{2}\theta$.

32. Solve the equation $7 \sin \theta + 4 \cos \theta = 1$ by substituting t-formulæ for sin θ and cos θ. [After substitution, multiply both sides by $1 + t^2$, arranging the equation as a quadratic in t. Two values of t give two values of $\frac{1}{2}\theta$ and so of θ.]

33. As in Ex. 32, solve the equation $3 \cos \theta - 4 \sin \theta = 1$.

[See pp. 131, 132 for another, and usually better, way of solving this type of equation.]

a cos θ + b sin θ

Expressions of this kind occur often enough to justify special study. In Mechanics it is sometimes necessary to find the greatest value of such an expression when θ is varied, or to find the value of θ for which it has a given value.

Ex. 14 of the last set of examples has shown that $5 \sin (\theta + \alpha) = 4 \sin \theta + 3 \cos \theta$, where $\tan \alpha = \frac{3}{4}$ and that $5 \cos (\theta - \alpha) = 4 \cos \theta + 3 \sin \theta$. It looks therefore as if such an expression as $4 \cos \theta + 7 \sin \theta$ could be put in either of the forms $p \sin (\theta + \beta)$ or $p \cos (\theta - \beta)$.

The same suggestion can be obtained by drawing a graph.

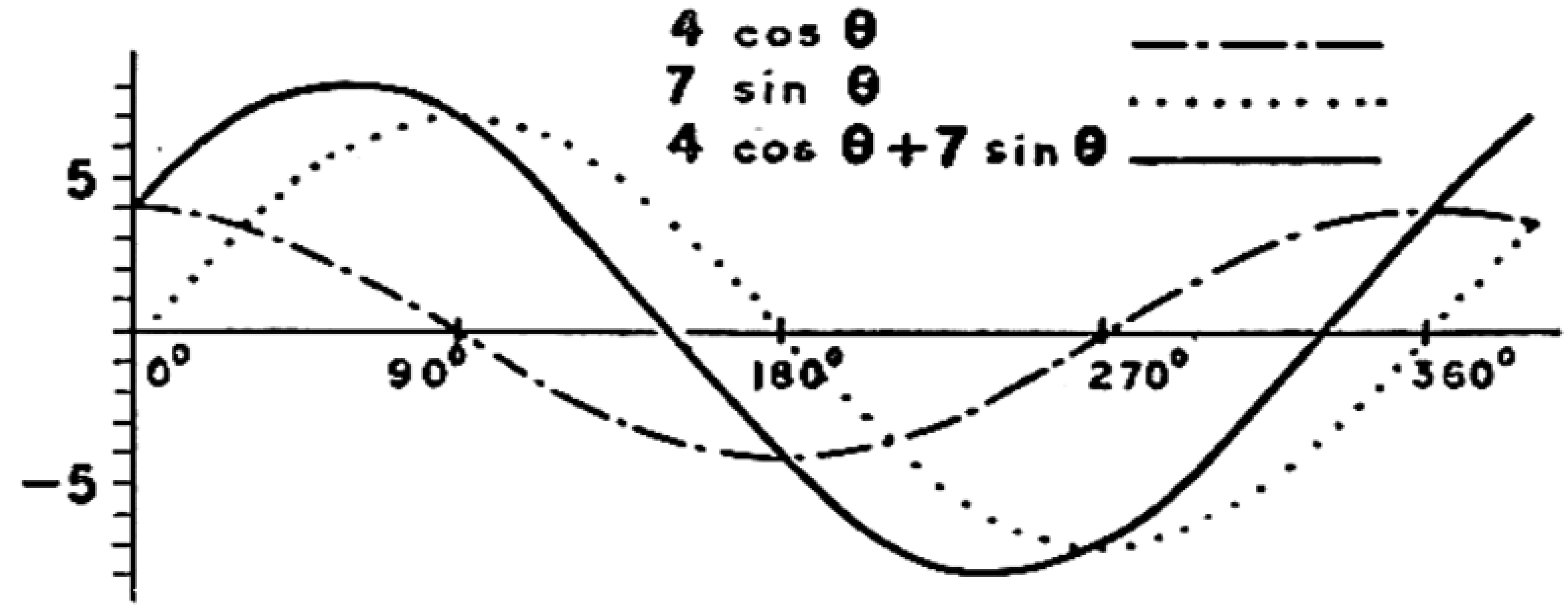

FIG. 135.

The graph appears to be a sine (or cosine) curve with a greatest height (and depth) of about 8 but with a start of about 30° on sin θ. This suggests that $4 \cos \theta + 7 \sin \theta$ is roughly equal to $8 \sin (30^\circ + \theta)$ which when expanded is $8 \sin 30^\circ \cos \theta + 8 \cos 30^\circ \sin \theta$. Here $8 \sin 30^\circ$ is accurately 4 but $8 \cos 30^\circ$ is not accurately 7 being 6·9.

To get it right, put $r \sin \alpha = 4$ and $r \cos \alpha = 7$, giving $\tan \alpha = 4/7$, and Fig. 136 shows that

$$r^2 = 4^2 + 7^2 = 16 + 49 = 65.$$

So $r = \sqrt{65} \simeq 8{\cdot}062$ while $\alpha = \tan^{-1} 4/7 = 29°44'$. Hence $4 \cos \theta + 7 \sin \theta = 8{\cdot}062 \sin (29°44' + \theta)$.

Since the maximum value of a sine is 1, the maximum value of the expression is 8·062 and this is reached when

$$29°44' + \theta = 90°, \textit{ i.e., } \theta = 60°16'.$$

Examples—58

1. Use Fig. 136 to show that

$$4 \cos \theta + 7 \sin \theta = \sqrt{65} \cos (\theta - \beta),$$

where $\tan \beta = 7/4$.

[This form for the expression is often convenient in solving equations.]

2. If $\alpha + \beta = 90°$ prove that

$$\sin (\alpha + \theta) = \cos (\beta - \theta) = \cos (\theta - \beta).$$

3. (i) Show that $3 \sin x + 4 \sin (x + 60°) = \sqrt{37} . \sin (x + \beta)$ where $\tan \beta = 2\sqrt{3}/5$. (ii) Show that $a \sin (x + \alpha) + b \sin (x + \beta)$ may be expressed in the form $c \sin (x + \gamma)$.

[Thus the sum of two sine waves of the same wavelength but differing in amplitude and phase is another sine curve of the same wavelength.]

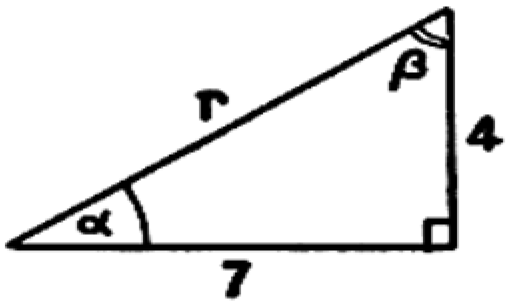

FIG. 136.

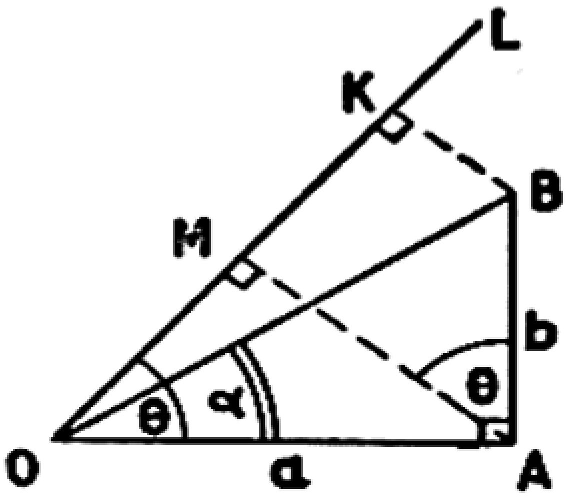

FIG. 137.

The Equation a cos θ + b sin θ = c

This equation implies that, as shown in Fig. 137, if OL makes θ with OA and AB is perp. to OA, then

Projection of OA on OL + projection of AB on OL = c.

These projections are OM and MK, so OK must equal c.

If $\angle OAB = \alpha$ and $OB = r$ then $\angle BOK = \theta - \alpha$ and $OK = r \cos (\theta - \alpha)$.

$\therefore\ r \cos (\theta - \alpha) = a \cos \theta + b \sin \theta.$

The figure shows that $r = \sqrt{(a^2 + b^2)}$ and that

$$r \cos \alpha = a, \ r \sin \alpha = b.$$

Examples—59

[Ex. 1 to 4 on Fig. 137.]

1. Taking the equation as $4 \cos \theta + 3 \sin \theta = 4{\cdot}5$, *i.e.*, taking OA = 4, AB = 3, OK = 4·5, show that OB = 5, $\alpha = 36°52'$, $\angle$BOK $= 25°50'$ so that one solution of the equation is $\theta - 36°52' = 25°50'$, *i.e.*, $\theta = 62°42'$.

2. *Geometrical Construction.*

Show (i) that K lies on the circle on OB as diameter (the circle OAB); (ii) that the equation may be solved by drawing the circle OAB and placing in it a chord OK of length 4·5; draw this figure accurately; (iii) that there are two such chords, OK, OK′, one on each side of OB, making equal angles with OB; (iv) that if $\angle$AOK′ $= \theta$ then $5 \cos (\alpha - \theta) = 4{\cdot}5$ (note that $\cos (\alpha - \theta) = \cos (\theta - \alpha)$ by the $-\theta$ formula). Deduce that a second solution of the equation is $36°52' - 25°50' = 11°2'$.

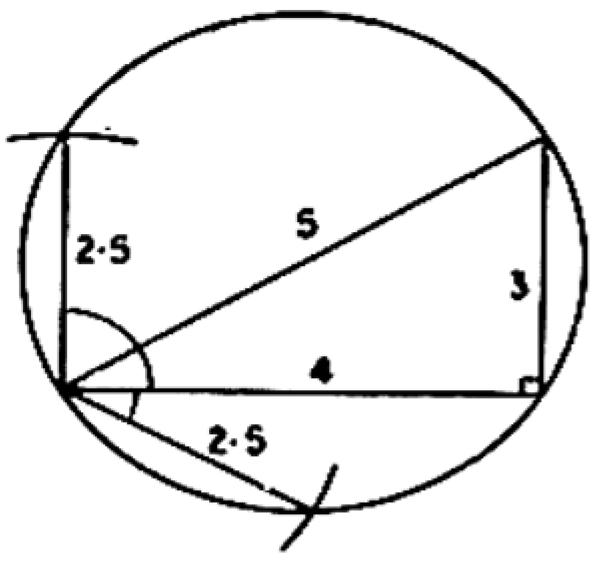

FIG. 138.

3. Show by drawing Fig. 138 accurately that the solutions of $4 \cos \theta + 3 \sin \theta = 2{\cdot}5$ are one obtuse and the other negative, and prove that they are given by $\theta = 36°52' \pm 60°$.

4. If in Fig. 137 OA = 3, AB = 2, find OB. Express $3 \cos \theta + 2 \sin \theta$ in the form $r \cos (\theta - \alpha)$ and find two solutions of $3 \cos \theta + 2 \sin \theta = 3{\cdot}2$.

5. If $5 \cos \theta + 12 \sin \theta = r \sin (\alpha + \theta)$, find r and α.

6. Find r and α in the following :—

(i) $\cos \theta + 2 \sin \theta = r \sin (\alpha + \theta)$
(ii) $3 \cos \theta - 4 \sin \theta = r \sin (\alpha - \theta)$
(iii) $2 \cos \theta - 3 \sin \theta = r \cos (\alpha + \theta)$
(iv) $8 \cos \theta + 15 \sin \theta = r \cos (\alpha - \theta)$
(v) $3 \sin \theta - \cos \theta = r \sin (\theta - \alpha)$.

7. Show that the greatest value of $a \cos \theta + b \sin \theta$ is $\sqrt{(a^2 + b^2)}$, and that it occurs when $\tan \theta = b/a$.

8. If a box weighing W lbs. is dragged along a level floor by means of a rope inclined at θ to the floor, the pull in the

rope is P lb., where $P = \mu W/(\cos\theta + \mu\sin\theta)$, μ being the coefficient of friction. If $\mu = \cdot 5$ and $W = 20$, find (i) the value of θ when $P = 12$; (ii) the least value of P and the value of θ for which this occurs. [Find when $\cos\theta + \mu\sin\theta$ is greatest.]

9. A wardrobe is 6 ft. high and 8 ft. wide. One end is raised until one of the top corners of the wardrobe touches the ceiling, which is 9 ft. above the floor. Through what angle is the wardrobe turned? See Fig. 139.

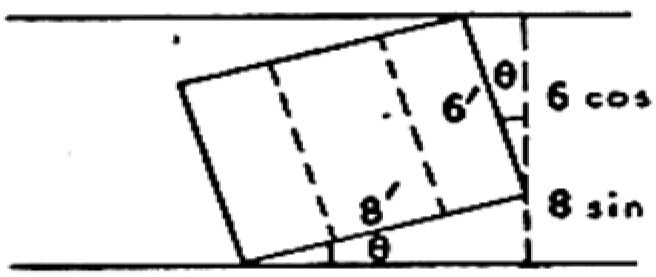

FIG. 139.

10. Solve $7\cos\theta + 6\sin\theta = 8$.
(i) by drawing the graph of $7\cos\theta + 6\sin\theta$ from 0° to 90°.
(ii) by transforming the equation to $r\cos(\theta - \alpha) = 8$ where $\tan\alpha = 6/7$. (iii) By using the t-formulæ $\sin\theta = 2t/(1 + t^2)$, $\cos\theta = (1 - t^2)/(1 + t^2)$.

11. Solve the following equations by the (r, α) method (0° to 360°): (i) $6\cos\theta + 8\sin\theta = 5$; (ii) $4\sin\theta - 3\cos\theta = 2$; (iii) $12\sin\theta - 5\cos\theta = 7$; (iv) $15\cos\theta + 8\sin\theta = 10$; (v) $2\sin\theta + 3\cos\theta = 3{\cdot}5$.

12. Solve the following equations by the t-formulæ method: (i) $2\cos\theta - 11\sin\theta = 10$; (ii) $\cos\theta + 7\sin\theta = 5$; (iii) $7\cos\theta - 4\sin\theta = 8$; (iv) $7 = 11\tan\theta + 13\sec\theta$ [multiply by $\cos\theta$]; (v) $\cos\theta + \sin\theta = 1{\cdot}2$.

Examples—60 : Miscellaneous

Examples on the Addition Theorems

1. Prove the following identities : Ex. 1 to 6 :—

(i) $\cot\alpha + \tan\alpha = 2\operatorname{cosec} 2\alpha$; (ii) $\cot\alpha - \tan\alpha = 2\cot 2\alpha$; (iii) $\operatorname{cosec} 2\alpha + \cot 2\alpha = \cot\alpha$.

2. (i) $(\sin\alpha + 3\cos\alpha)^2 = 3\sin 2\alpha + 4\cos 2\alpha + 5$;
(ii) $\cos^4\alpha - \sin^4\alpha = \cos 2\alpha$.

3. (i) $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta}$;

(ii) $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}$.

4. (i) $(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)=\sin(\alpha+\beta)+\cos(\alpha-\beta)$.
(ii) $(\sin\alpha-\cos\alpha)(\sin\beta-\cos\beta)=\cos(\alpha-\beta)-\sin(\alpha+\beta)$.

5. $16\sin\alpha\cos\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha=\sin 16\alpha$.

6. If $x=\cos\theta+\cos 2\theta$, $y=\sin\theta+\sin 2\theta$, prove that
(i) $x^2+y^2=2+2\cos\theta$;
(ii) $x^2-y^2=\cos 2\theta+2\cos 3\theta+\cos 4\theta$;
(iii) $2xy=\sin 2\theta+2\sin 3\theta+\sin 4\theta$.

Formula Recognition

The expressions of Ex. 7–10 can be simplified by recognising the formulæ which they resemble. The terms may not be in the usual order :—

7. (i) $\sin\alpha\sin\beta+\cos\alpha\cos\beta$; (ii) $\sin\alpha\cos 2\alpha+\cos\alpha\sin 2\alpha$; (iii) $\cos^2\alpha-\sin^2\alpha$.

8. (i) $2\sin(\alpha+\beta)\cos(\alpha+\beta)$; (ii) $\cos\alpha\sin\alpha$; (iii) $\sin^2 2\alpha-\cos^2 2\alpha$.

9. (i) $\sin(\alpha+\beta)\cos\gamma+\cos(\alpha+\beta)\sin\gamma$.
(ii) $\cos(\alpha-\beta)\cos\beta-\sin(\alpha-\beta)\sin\beta$.

10. (i) $\dfrac{\tan 2\theta-\tan\phi}{1+\tan 2\theta\tan\phi}$. (ii) $\dfrac{2\tan 2\theta}{1-\tan^2 2\theta}$.

11. (i) By adding the "expanded forms" of $\sin(\theta+\phi)$ and $\sin(\theta-\phi)$, taking $\theta>\phi$ show that

$$2\sin\theta\cos\phi=\sin(\theta+\phi)+\sin(\theta-\phi).$$

(ii) If $\theta+\phi=A$ and $\theta-\phi=B$, show that $\theta=\frac{1}{2}(A+B)$ and find the corresponding value of ϕ.

(iii) Write the result of (i), with right-hand side first, in terms of A and B.

12. Use the "expanded forms" of $\cos(\theta+\phi)$ and $\cos(\theta-\phi)$, taking $\theta>\phi$, exactly as in parts (i), (ii), (iii) of Ex. 11.

13. Repeat Ex. 11, 12, using $\sin(\theta+\phi)-\sin(\theta-\phi)$ and $\cos(\theta-\phi)-\cos(\theta+\phi)$.

[The alteration of the order in the last expression is because the smaller angle has the larger cosine.]

Note. The formulæ of Ex. 11, 12, 13 are important for the mathematical specialist. So much so, that it has been said that "without them no further advance can be made."

Examples—61 : "Advanced Level" Questions

1. The sides of a triangle are 5 in., 6 in., 8 in. Calculate the largest angle. Calculate also the radius of the circumscribing circle.

2. A, B, C are consecutive milestones on a straight road AX running from west to east and T is a town to the north of the road. A surveyor measures the angles TAX, TBX and TCX. The record for $\angle$TBX is lost, but it is known that $\angle$TAX is $34°30'$ and $\angle$TCX is $79°$. Recover the record for $\angle$TBX by calculation, finding first the distance of the town from each of the milestones.

3. (i) Prove that $\sin 3(x + \frac{1}{2}\pi) = - \cos 3x$, x being the radian measure, and give similar equivalents for $\cos 3(x+\frac{1}{2}\pi)$ and $\tan 3(x + \frac{1}{2}\pi)$.

(ii) Give all the solutions between 0° and 360° of the equation $\sec x = 2{\cdot}2$.

4. Prove the formula $\sin (A+B)=\sin A \cos B+\cos A \sin B$ and write down the corresponding formula for $\cos (A - B)$. Express $\sin 3\theta$ in terms of $\sin \theta$ and verify your formula by putting $\theta = 45°$.

5. Assuming formula for $\sin (A + B)$ and $\cos (A + B)$, prove that $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

If the angle A of a triangle is 45° and a perpendicular from the point A on the opposite side BC divides BC into segments of lengths x, y, prove that

$$ap = p^2 - xy$$

when $a = \text{BC}$ and p is the length of the perpendicular.

6. Quote the formula for $\cos (A + B)$ and prove that

(i) $\cos 2A = 1 - 2 \sin^2 A$
(ii) $\cos 4A = 1 - 8 \sin^2 A + 8 \sin^4 A$.

Explain a method of calculating $\sin 7\frac{1}{2}°$ by using one or more of these formula.

7. ABC is a triangle in which $AB = BC = 1$ and $\angle B=2\theta$; N bisects AC and AL,NK are drawn perpendicular to BC. Use this figure to prove that $1 - \cos^2 \theta = \cos^2 \theta - \cos 2\theta$.

8. In any triangle ABC proves that $a = 2\text{R} \sin \text{A}$, the side a being opposite that angle A (which may be acute or obtuse) and R being the radius of the circumscribing circle. If the side c and the acute $\angle \text{B}$ are given, show that the side b is least if the triangle is right angled.

9. Express $2 \sin x - 3 \cos x$ in the form $r \sin (x - \alpha)$ and find the two smallest positive angles which make

$$2 \sin x - 3 \cos x = 3.$$

Sketch the graph of the expression from 0° to 360°.

10. Draw the graph of $2 \cos 2x$ from $x = 0°$ to $90°$and drawing as much of the graph of $\tan x$ as is needed, find the least positive root of the equation $2 \cos 2x = \tan x$.

11. Draw the graph of $y = \cos x° + \frac{1}{2} \cos 2x°$ between $x = 0°$ and $x = 80°$ and determine approximately the value of x for which $y = 0$. (Use a scale to suit the fact that the values of y lie between $-\frac{1}{2}$ and $+1\frac{1}{2}$).

12. Show that $\cos x - \sin x = \sqrt{2} \cos (x + \frac{1}{4}\pi)$.

Draw the graph of $y = \cos x - \sin x$ from $x = -2\pi$ to $x = 2\pi$ and use it to obtain the numerically least positive and negative angles satisfying $\cos x - \sin x = \frac{1}{8}$.

13. Write down the expressions for $\cos (\theta + \phi)$ and $\cos (\theta - \phi)$. If $\cos \theta \cos \phi = \frac{1}{2}$ and $\sin \theta \sin \phi = \frac{1}{4}$, find θ and ϕ (acute).

14. Prove (i) $2 \sin (45° + \theta) . \cos (45° - \theta) = 1 + \sin 2\theta$
(ii) $2 \cos (45° + \theta) . \cos (45° - \theta) = \cos 2\theta$
(iii) $\tan (45° + \theta) . \tan (45° - \theta) = 1$.

15. If $\sin \alpha = \frac{3}{5}$, $\sin \beta = \frac{5}{13}$, $\sin \gamma = \frac{7}{25}$, find $\sin (\beta + \gamma)$, $\cos (\beta + \gamma)$, $\sin (\alpha + \beta + \gamma)$, $\cos (\alpha + \beta + \gamma)$, $\tan (\alpha + \beta + \gamma)$.

Examples—62 : More Trigonometrical Equations

Set A involve the addition formulæ and the double-angle formulæ. In Nos. 1 to 8 give answers from 0° to 180° inclusive.

A. 1. $5 \sin 2x = \sin x$.
[Hint : $10 \sin x \cos x - \sin x = 0$, giving
$\sin x (10 \cos x - 1) = 0 \quad \therefore \sin x = 0$ or $\cos x = \cdot 1$.]

2. $4 \sin 2x = 2 \cos x$.

3. $\cos 2x + \cos x = 0$. [Quadratic in $\cos x$.]

4. $\cos 2x + \sin x = 0$. [Quadratic in $\sin x$.]

5. $3 \cos 2x + \cos x + 1 = 0$.

6. $\sin 2x = 3 \sin^2 x$.

7. $\cos 2x = \cos^2 x$.

8. $\tan x = 3 \sin x$. [Multiply both sides by $\cos x$.]

In Nos. 9 to 14 give answers 0° to 90° inclusive.

9. $\cos (x + 30°) = \sin x$.

[Sol'n. $\cos x \cos 30° - \sin x \sin 30° = \sin x$. Dividing by $\cos x$, this gives $\cos 30° - \tan x \,.\, \sin 30° = \tan x$, so that $\tan x = \cos 30°/(1 + \sin 30°)$].

10. $3 \sin (x + 20°) = 2 \cos x$.

11. $\sin (x + 30°) = 1{\cdot}5 \sin x$.

12. $3 \sin 2x + 4 \cos^2 x = 1$.

[Hint : $6 \sin x \cos x + 4 \cos^2 x = \cos^2 x + \sin^2 x$.]

13. $\tan x + \cot x = 3$.

14. $\sin 2x - \cos 2x = \sin x (\sin x + \cos x)$.

Set B are solved by the intersections of two graphs.

B. 1. $\sin 2x = 1 - \cos x$.

[Scales 1″ for 20° ; 5″ for one unit. Graph 0° to 90°.]

2. $\cos 2x = \frac{1}{2} \tan x$.

[Scales 1″ for 10° ; 5″ for one unit. Graph 15° to 60°.]

3. $3 \sin \frac{1}{2}x = 2 - 2 \cos 2x$. Graph from 0° to 90°.

4. Draw the graphs of $\frac{1}{4}x°$ and $\tan 10x°$ from $x = 0$ to $x = 6$, and use them to solve the equation $\frac{1}{4}x = \tan 10x°$.

5. Tabulate the values of $\sin x$ when $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and draw the graphs of $\sin x$ and $\frac{3}{4}x$.

Solve the equation $\sin x = \frac{3}{4}x$, giving the value of x in radians.

[Scales, 6″ for π radians ; 5″ for one unit of $\sin x$, etc.]

6. Draw the graphs of $\pi \sin^2 x$ and $2x$, from 0 to $\frac{1}{2}\pi$ radians, and hence solve the equation $2x = \pi \sin^2 x$, giving the angles in radians. Take 6″ for π radians and for π units on the vertical axis. Use the values of x tabulated in No. 5.

7. The area of a segment of a circle of radius r is $\frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta$, if θ radians is the angle subtended at the centre of the circle. Find θ if the area of the segment is one-third of that of the circle. [This gives the equation $\frac{1}{3}\pi r^2 = \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta$, or $\theta - \sin \theta = \frac{2}{3}\pi$. Draw the graphs of $\theta - \frac{2}{3}\pi$ and $\sin \theta$ from 0 to π.]

8. Find the angle of a sector of a circle in which the chord bisects the area of the sector. [Area of sector $= \frac{1}{2}r^2\theta$, area of triangle formed by radii and chord $= \frac{1}{2}r^2 \sin \theta$. Hence $\sin \theta = \frac{1}{2}\theta$. Solve graphically.]

[It is usually quicker and more accurate to tabulate values, after only sketching the graphs. See also p. 166.

Thus to solve the equation $\theta - \sin \theta = \frac{2}{3}\pi$ of Ex. 7 write it $\theta - \frac{2}{3}\pi = \sin \theta$, remember $\frac{2}{3}\pi = 120°$ and $\sin 150° = \sin 30°$, etc.

Degrees	140°	150°	148°	149°	149°15′	16′	17′
$\theta - \frac{2}{3}\pi$	·3491	·5236	·4887	·5061	·5105	·5107	·5109
$\sin \theta$	·6428	·5000	·5299	·5150	·5113	·5110	·5108
	Diff.	+ ·0236	− ·0412	− ·0089	·0008		

Thus the angle is 149°17′ to nearest minute; $\theta = 2{\cdot}0944$].

CHAPTER VIII

ENDINGS AND BEGINNINGS

Calculation of Tables

You have been using trigonometrical tables all through this book. Have you ever wondered how these tables are calculated ?

The results of the previous chapter show how it *can* be done, though the ways suggested in the following examples would be supplemented or replaced by others in order to construct a complete set of tables.

Examples—63

[Answers to be checked by comparison with the tables.]

1. Turn 1° into radians to 5 dec. pl., and use the approximations (see Ex. 25, p. 113) $\sin\theta \bumpeq \theta$ and $\cos\theta \bumpeq 1 - \frac{1}{2}\theta^2$ to find sin 1° and cos 1°.

2. Explain, by means of a diagram, why $\sin 30^\circ = \frac{1}{2} = \cos 60^\circ$, and $\cos 30^\circ = \frac{1}{2}\sqrt{3} = \sin 60^\circ$, and obtain decimal values for the latter, taking $\sqrt{3} \bumpeq 1{\cdot}7321$.

3. Use the addition formulæ (p. 124) to calculate sin 61°, sin 59°, cos 61°, cos 59°.

[$\sin 61^\circ = \sin(60^\circ + 1^\circ) = \sin 60^\circ \cos 1^\circ + \cos 60^\circ \sin 1^\circ$.]

4. Use the formula $\cos 2\theta = 2\cos^2\theta - 1$, rearranged as $\cos\theta = \sqrt{\frac{1}{2}(1 + \cos 2\theta)}$, to obtain in succession cos 30°, cos 15°, cos $7\frac{1}{2}$°, cos $3\frac{3}{4}$°, starting from $\cos 60^\circ = \frac{1}{2}$.

5. Use $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$ to work from the sine and cosine of 1° (No. 1 above) to 2°, 4°, 8°, 16°.

[The work becomes rather troublesome ; here is a sample of it :

Having found $\sin 8^\circ = {\cdot}1392$, $\cos 8^\circ = {\cdot}9903$

$$\begin{array}{lll} \text{then } \sin 16^\circ = 2 \times {\cdot}1392 \times {\cdot}9903 & {\cdot}3010 & \\ \qquad\qquad = {\cdot}2756 & \bar{1}{\cdot}1436 & \bar{2}{\cdot}2872 \\ \text{and } \cos 16^\circ = 1 - 2\,({\cdot}1392)^2 & \bar{1}{\cdot}9957 & ({\cdot}01937) \\ \qquad\qquad = {\cdot}9613. & \overline{\bar{1}{\cdot}4403} & {\cdot}03874] \end{array}$$

6. Use the sine and cosine of 16° to find tan 16°, cot 16°, sec 16°, cosec 16°. Of course we ought to work with at least five figures to make reasonably sure of agreeing with four-figure tables, but the method is the chief thing here.

Interpolation

7. Suppose we knew sin 60° (·8660) and sin 63° (·8910), and attempt to calculate the sines of the angles from 60° to 63° in steps of 6′. The gain in the sine is ·0250 for 3° or 180′, which is $\cdot 0008\frac{1}{3}$ for 6′. Hence we should get $\sin 60^\circ 6' = \cdot 8668\frac{1}{3}$ $\sin 60^\circ 12' = \cdot 8677\frac{2}{3}$. Continue, and note how far behind the tables your values are, especially in the middle of the 61° line. This is because you have taken the sine graph from 60° to 63° as straight instead of a convex curve. (See **Fig.** 140.)

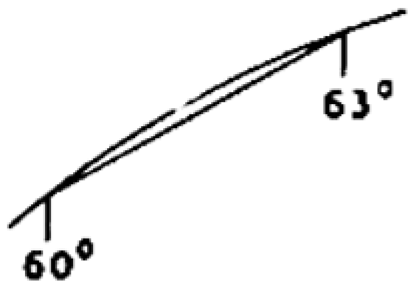

FIG. 140.

Differences

8. Since 1 radian $\simeq$ 3438′, 1′ = 1/3438 radian.

If m is less than 6 we can take $\sin m' = \dfrac{m}{3438}$ and $\cos m' = 1$ without affecting even the fifth decimal place. Show that $\sin(\theta^\circ + m') \simeq \sin\theta^\circ + \dfrac{m}{3438}\cos\theta^\circ$, and use this to calculate the numbers in the difference column suitable for angles on the 60° line. [The difference for m' is $\sin(\theta^\circ + m') - \sin\theta^\circ$ which in this case is $\dfrac{m}{3438}\cos 60^\circ = \dfrac{m \times \cdot 5}{3438} = \cdot 000146m$]. Calculate also the differences for 1′, 2′, 3′, 4′, 5′ on the 61° line of the sine table giving them as 1·46, etc. Compare with the tables.

PROJECTION OF FIGURES

The orthogonal projection of a point on to a plane is the foot of the perpendicular from the point to the plane.

In Fig. 141 P, Q, R, S are the projections of A, B, C, D on to a plane β.

The projections of the straight lines AB, BC, CD, DA are the straight lines PQ, QR, RS, SP.

If ABCD is a rectangle in a plane α which meets the plane β in the line MN, and if AD$\parallel$MN, then PQRS is also a rectangle.

The lines BAM, QPM are both at right angles to MN, so that the angle AMP ($= \theta$) is the angle between the planes α and β.

If the sides AB, BC are of lengths l, b respectively, then the lengths of PQ,QR will be $l \cos \theta$, b (since BCRQ is a rectangle).

$\therefore$ The areas of the rectangles ABCD, PQRS are lb, $lb \cos \theta$.

$\therefore$ Projected area = original area $\times \cos \theta$.

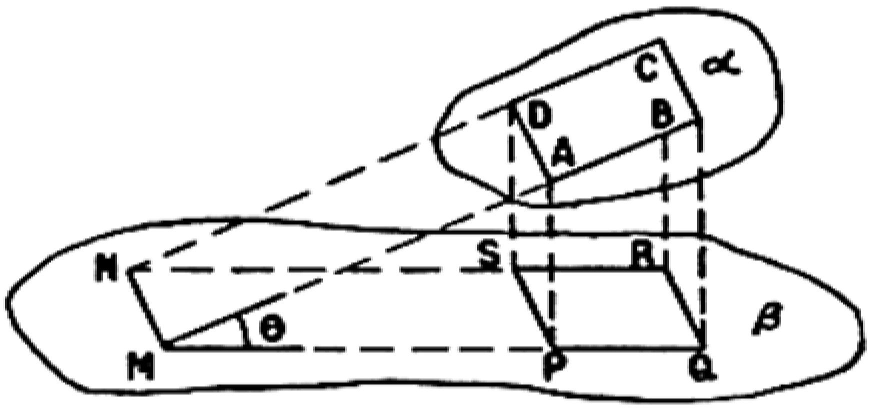

FIG. 141.

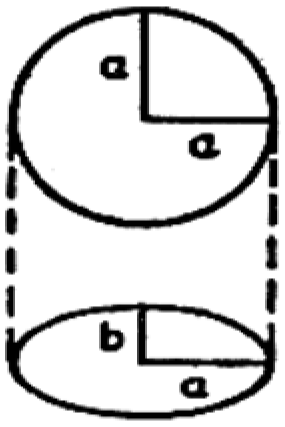

FIG. 142.

Although we have proved this only for a rectangle in a special position, the result is true for a figure of any shape, but the formal proof is too long to be given here.

It should be noted that as a result of projection :—

(i) The lengths of lines parallel to the common line MN are unaltered.

(ii) The lengths of lines perpendicular to the common line MN are multiplied by $\cos \theta$.

Example. If a line of length l in plane α makes an angle ϕ with MN, prove that its projection on plane β is of length $\sqrt{(l^2 \cos^2 \phi + l^2 \sin^2 \phi \cos^2 \theta)} = \sqrt{(l^2 - l^2 \sin^2 \theta \sin^2 \phi)}$.

The Area of an Ellipse (Fig. 142)

If a circle in the plane α is projected on to the plane β the result is an *ellipse*. If the circle has radius a, then the major semi-axis of the ellipse will be a, and will be parallel to MN. The minor semi-axis of the ellipse will be of length $a \cos \theta$, and will be at right angles to MN. As the area of circle is πa^2, that of the ellipse will be $\pi a^2 \cos \theta$. If the minor semi-axis is denoted by b, as is customary, then $a \cos \theta = b$, and the area $\pi a^2 \cos \theta = \pi a \,.\, a \cos \theta = \pi ab$.

The angle θ between the planes is then $\cos^{-1} b/a$.

Oblique Sections

In Fig. 141, ABCD may be regarded as an oblique section of a brick (or right prism) whose base or cross section is PQRS. Since the areas of ABCD and PQRS are as $1 : \cos\theta$, the area of any plane section of a prism or cylinder is to the area of cross section as $1 : \cos\theta$ or $\sec\theta : 1$.

Examples—64

1. If in Fig. 141 the areas of the two rectangles are 50 sq. in. and 40 sq. in. find the value of θ.

2. If in Fig. 141 the lengths of PQ and QR are 10 in. and 6 in., and $\theta = 40°$, find the lengths of AB and BC.

3. A solid cylindrical shaft of diameter 4 in. is sawn through obliquely. What is the area of surface exposed if the cut makes (i) 45°, (ii) 60° with the axis of the cylinder ?

4. A block of wood has a rectangular cross-section PQRS, and PQ, QR are 4″, 3″ respectively. An oblique saw-cut is made across it so that the edges PA, QB, SD are 6″, 9″, 10″ long. Calculate, using the construction suggested in Fig. 143, (i) the lengths AB, AD, BD ; (ii) the area of △PQS ; (iii) the area of △ABD ; (iv) the angle between planes ABCD, PQRS ; (v) the length RC.

[Hint for (v) : ABCD is a par'm since (*a*) it is a plane figure, (*b*) its opposite sides are in parallel planes. ∴ AC, DB have same mid-point.]

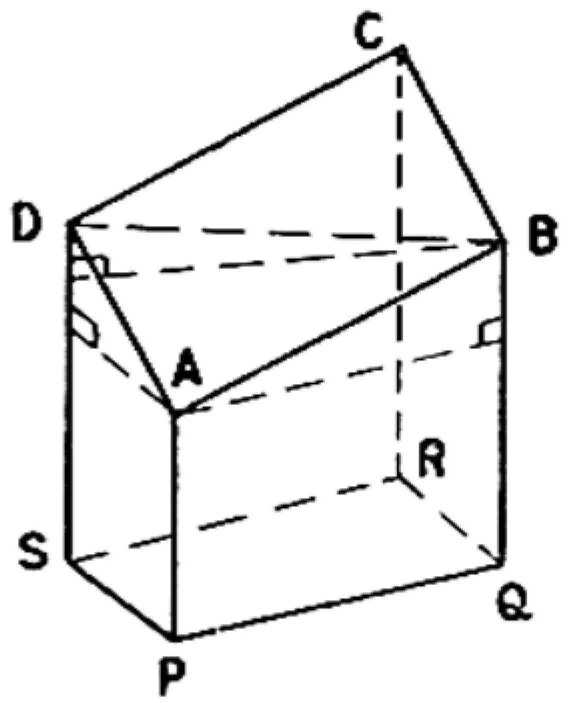

FIG. 143.

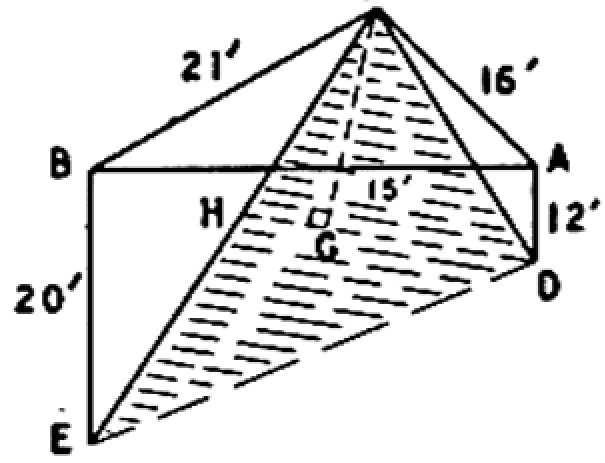

FIG. 144.

5. A coal seam crops out at C, and its depths below A, B, two points at the same level as C, are 12 ft. and 20 ft. If AB, BC, CA are 15 ft., 21 ft. and 16 ft., calculate (i) the sides

of $\triangle CDE$, (ii) the areas of triangles ABC, CDE, (iii) the angle between the coal seam (assumed plane) and the horizontal (Fig. 144).

6. Check the answer to No. 5 in the following way: draw a horizontal line through D to cut CE at H. Since DH is a horizontal line in the plane CDE, the line CG perpendicular to DH is a line of greatest slope. The angle between CG and the horizontal is the required angle. The length CE is known. The depths of H and E are 12 ft. and 20 ft., so that $CH : CE = 12 : 20$. (i) From $\triangle CDE$ calculate angle DCE. (ii) From $\triangle DCH$ calculate angle DHC. (iii) From CH and angle CHG calculate CG. The depth of G below the ground is 12 ft., so that the angle required is $\sin^{-1} 12/CG$.

7. An equilateral triangle ABC of side 60 ft. is marked out on level ground. Borings AX, BY, CZ made vertically at A, B, C show a coal seam at depths 9, 89 and 100 ft. respectively. Find the angle between the seam and the horizontal.

8. In No. 5 find where the seam will crop out in the direction of BA, and, if CA is due East, the bearing of the "line of *strike*," *i.e.*, the line where the seam crops out. [If the seam meets BA produced at X, CX is the line of strike.]

9. A quarry (Fig. 145) has a horizontal floor and a vertical face. A mineral seam crops out along BA in the floor and along BT in the vertical face.

$$\angle TPB = 90° = \angle PQB = \angle TQB.$$

If $\angle TBP = 60°$ and $\angle PBQ = 40°$ find the *dip* of the seam, *i.e.*, the inclination to the horizontal. [TQ is a line of greatest slope. It is the hillside problem again.]

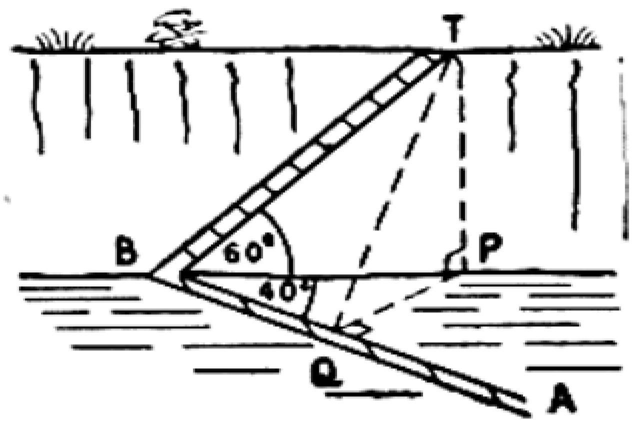

FIG. 145.

VECTORS

Vector Addition

There is a very important sense in which OC, Fig. 146, is thought of as the sum of OA and OB, these being the sides and OC the diagonal of a parallelogram.

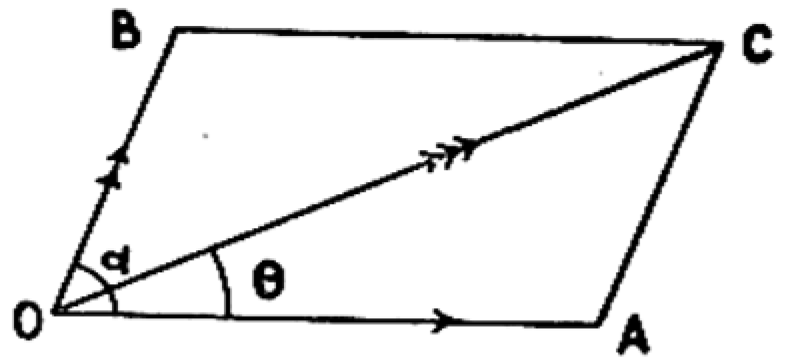

FIG. 146.

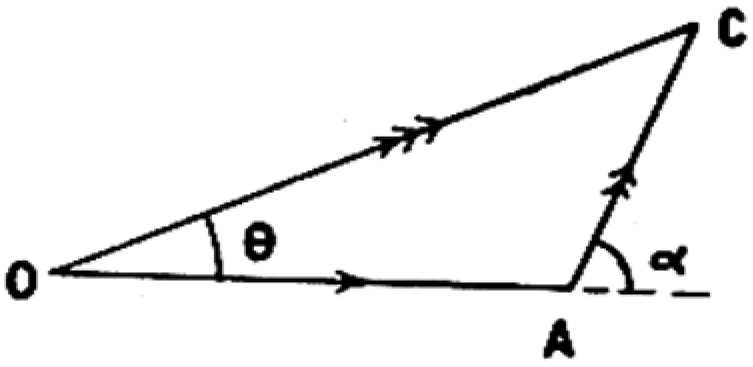

FIG. 147.

This is when OA and OB represent quantities called *vectors*,* examples of which are *displacements* (*i.e.*, distances moved), *velocities* and *forces*, all of which quantities have both magnitude and direction.

OC is also thought of as the sum of OA and AC or of OB and BC, for equal and parallel lines represent the same vector. This kind of adding is called " compounding by the parallelogram law " or simply " vector addition." The sum is often called the *resultant* of the vectors.

The resultant (or sum) of two or more vectors is usually found by drawing, calculation being only used in cases where drawing would not be accurate enough.

In the drawing, the parallelogram is very seldom used, only half of it being drawn, as in Fig. 147. This is called the vector triangle and in it the second vector AC starts at the point when the first vector OA finishes.

Notation for Vectors

To make sure that a vector is meant, write $\overline{\mathrm{OA}}$ for the vector whose length is OA, or of course you may write " vector OA."

Thus $\overline{\mathrm{OA}} + \overline{\mathrm{AC}} = \overline{\mathrm{OC}}$, while $\mathrm{OA} + \mathrm{AC} > \mathrm{OC}$ where these are lengths. In books about vectors it is not unusual to *print* them in clarendon (black) type, but this is not suitable for writing.

Resultant of Two Vectors

The trigonometry of Fig. 146 or Fig. 147 is quite simple. By the cosine formula

$$\mathrm{OC}^2 = \mathrm{OA}^2 + \mathrm{AC}^2 - 2\mathrm{OA}\,.\,\mathrm{AC}\cos\mathrm{OAC}$$

but $\angle\mathrm{OAC} = 180^\circ - \alpha \quad \therefore\ -\cos\mathrm{OAC} = \cos\alpha$

$$\therefore\ \mathrm{OC}^2 = \mathrm{OA}^2 + \mathrm{AC}^2 + 2\mathrm{OA}\,.\,\mathrm{AC}\cos\alpha.$$

---

* Vectors are much used in mechanics and in geometry. The examples of navigation (Chapter VI) on the sea or in the air are examples in the use of vectors. Vectors are also increasingly used in electricity, especially in dealing with alternating currents and with wireless.

Forces to behave strictly as vectors must all act at the same point. Forces acting in parallel lines on a rigid body do not have the same effect. Forces represented in magnitude, direction and line of action by OA, AC (Fig. 147) would have a resultant equal and parallel to OC but through A.

Thus if R is the resultant of vectors P, Q inclined at an angle α, we have (*note the + sign*)

$$R^2 = P^2 + Q^2 + 2PQ \cos \alpha$$

Also the sine formula gives for $\angle\theta$ between R and P

$$\sin \theta/Q = \sin \alpha/R.$$

N.B. The method given below for more than two vectors may be used for two vectors and is just as short as the above if θ is wanted as well as R.

Resultant of Three or More Vectors

When there are more than two vectors, the saving by not drawing parallelograms, but making each vector start where the previous one finishes, is much greater than with two vectors, and an altered method of calculation is used, as shown in the following :—

Worked Example

A ship sails from A to B, a distance of 50 *miles on a bearing* 70°, *then from B to C, a distance of* 40 *miles on a bearing* 10°, *and from C to D, a distance of* 30 *miles on a bearing* 300°.

What is the distance and bearing of D from A ?

This is really the problem of finding the resultant of three vectors. (See Figs. 148, 149, next page).

Draw lines AX, AY running east and north respectively.

If we can find how far D is north of AX and east of AY, we can calculate AD and the angle DAY.

If AM is the projection of AB on AX, then $\overline{AM}$ is the " easting " and $\overline{MB}$ the " northing " for the journey $\overline{AB}$.

The easting of AB is 50 cos 20° = 46·98
,, ,, ,, BC is 40 cos 80° = 6·94
,, ,, ,, CD is — 30 cos 30° = — 25·98

∴ the easting of D is 27·94 miles.

The northing of D is 50 sin 20° + 40 sin 80° + 30 sin 30°
= 17·10 + 39·39 + 15·00 = 71·49 miles.

$$\begin{aligned}\text{The distance AD} &= \sqrt{(27{\cdot}94^2 + 71{\cdot}49^2)}\\ &= \sqrt{(780{\cdot}4 + 5111)} = \sqrt{5891}\\ &= 76{\cdot}8 \text{ miles}\\ \text{The bearing } \theta &= \tan^{-1}(27{\cdot}94/71{\cdot}49)\\ &= 21^\circ 21'\end{aligned}$$

Logs.
1·4462
1·8542
$\bar{1}{\cdot}5810$

Resolving

In Fig. 148 $\overline{AM} + \overline{MB} = \overline{AB}$ and what we have done in the calculation above is to *replace* $\overline{AB}$ by these two vectors $\overline{AM}$ and $\overline{MB}$, which represent the distance gone eastward and the distance gone northward.

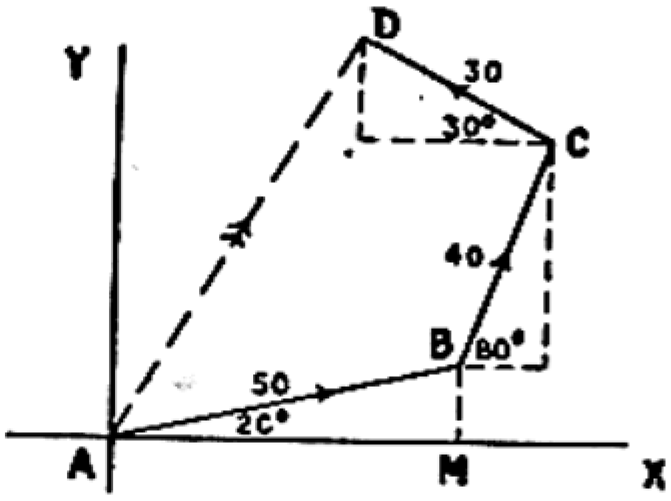

FIG. 148.

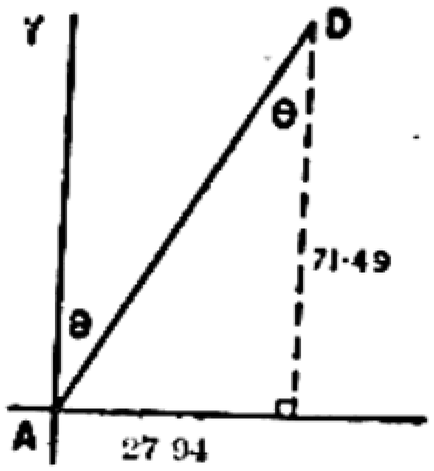

FIG. 149.

This process is called *resolving* the vectors in those two directions. $\overline{AM}$ and $\overline{MB}$ are called the *resolved parts* of AB. Similarly $\overline{BC}$ and $\overline{CD}$ have been resolved.

Resolved Parts and Components

If P, Q are two vectors whose resultant is R, then P and Q are called *components* of R.

If P and Q are at right angles they are called the *resolved parts*. The reason for this difference in names is that the resolved part P represents the *complete effect of R in the direction* of P, for Q in Fig. 151 has no effect either forwards or backwards along the line of P while in Fig. 150 it has an effect forward.

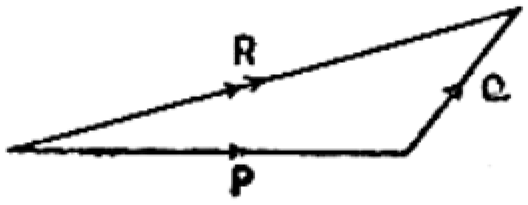

FIG. 150.

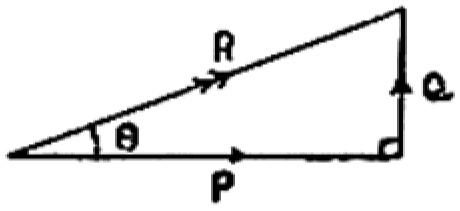

FIG. 151.

The resolved part of R at an angle of θ, Fig. 151, is R $\cos \theta$ and the resolved part perpendicular to this is R $\sin \theta$.

In the worked example $\overline{AM}$ represents the whole effect of $\overline{AB}$ in eastward motion. $\overline{MB}$, which is northward, neither helps nor hinders eastward motion.

A vector can be separated into two components in any two directions.

Worked Example

A 10-lb. weight is hung from a beam by two strings making 30° and 40° with the vertical. Find the pull exerted by each string. (Figs. 152, 153).

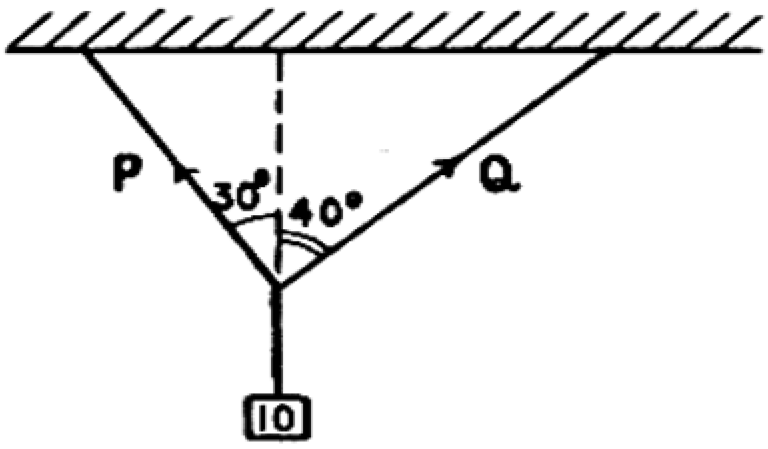

Fig. 152.

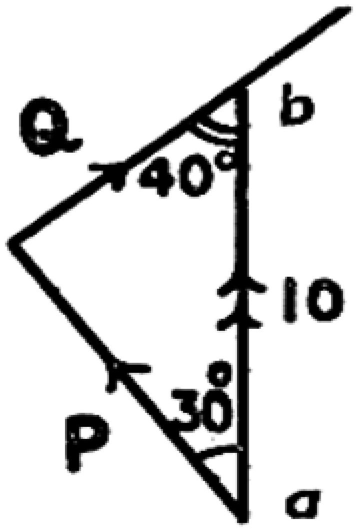

Fig. 153.

The combined pull in the strings must be 10 lb. vertically upward. If *ab* represents this force, lines drawn through *a* and *b* at angles of 30° and 40° give the components P and Q which are the pulls required. The problem is now easily solved by means of the sine formula or, of course, by measurement on an accurate drawing.

Examples—65

(Ex. 1, 2, 3, 4, 7 are intended for calculation but should be checked by accurate drawing.)

1. Find the resultant of two forces of 10 lb. and 4 lb. (i) if they make 50° with each other, (ii) if they make 130° with each other.

2. A force is of 10 lb. weight. Find (i) its resolved parts in directions making 25° and 65° with it, (ii) its components in two directions each of which makes 30° with it, (iii) its components if one makes 35° and the other 45° with it.

3. If a force of 39 lb. has two components 16 lb. and 25 lb., prove that the angle between them is $\cos^{-1} 4/5$.

4. What meaning can be given to dividing a vector into two components which make (for example) 25° and 35° with it *on the same side* ?

5. The resultant of two equal forces is equal to either of them. What is the angle between the two forces ?

6. In the vector triangle shown in Fig. 154, using the cosine and sine formula we get

$$R^2 = a^2 + b^2 + 2\,ab\cos\alpha \qquad \sin\theta = b\sin\alpha/R \qquad \text{(I)}$$

Using the method of resolved parts we get
$\mathrm{ON} = a + b\cos\alpha$; $\mathrm{NB} = b\sin\alpha$;

$$\tan\theta = b\sin\alpha/(a + b\cos\theta) \quad . \quad . \quad . \quad . \quad \text{(II)}$$

Show that these methods give the same results by using (II) to calculate R and sin θ.

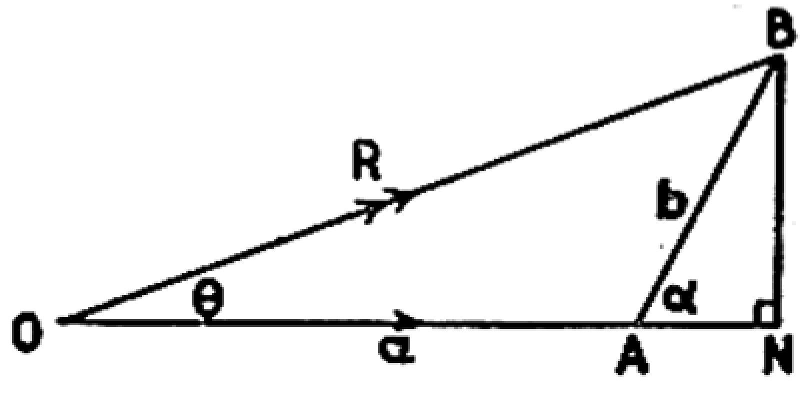

FIG. 154.

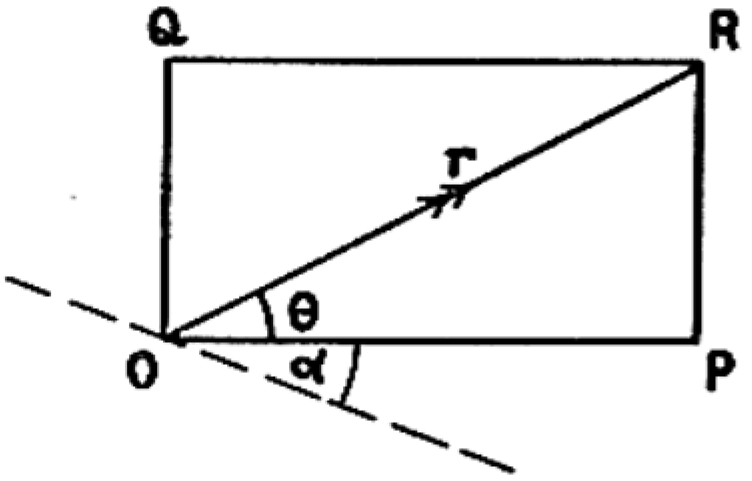

FIG. 155.

7. If one vector is half another what angles between them will make their resultant : (i) equal to the larger vector, (ii) equal to three-quarters of the larger vector ?

8. (i) In Fig. 155 the length of OR is r ; write down the lengths of OP, OQ in terms of r, θ.

(ii) Resolve $\overline{\mathrm{OP}}$ and $\overline{\mathrm{OQ}}$ each along and at right angles to $\overline{\mathrm{OR}}$ and show how this verifies that $\overline{\mathrm{OR}}$ is the resultant.

9. In Fig. 155 what are the resolved parts of $\overline{\mathrm{OP}}$, $\overline{\mathrm{OQ}}$, $\overline{\mathrm{OR}}$ along and perpendicular to the dotted line ? Verify that in each case the resolved part of $\overline{\mathrm{OR}}$ is the algebraic sum of the other two.

10. (i) If M is the mid-point of AB show that

$$\overline{\mathrm{OA}} + \overline{\mathrm{OB}} = 2\overline{\mathrm{OM}} ;$$

(ii) If K lies on AB so that AK = 2KB show that $\overline{\mathrm{OA}} + 2\overline{\mathrm{OB}} = 3\overline{\mathrm{OK}}$.

(iii) If G is the centre of gravity of ABC show that

$$\overline{\mathrm{OA}} + \overline{\mathrm{OB}} + \overline{\mathrm{OC}} = 3\overline{\mathrm{OG}}.$$

11. ABCD is a quadrilateral ; show that

(i) $\overline{\mathrm{AB}} + \overline{\mathrm{BC}} + \overline{\mathrm{CD}} + \overline{\mathrm{DA}} = 0$

(ii) $\overline{\mathrm{AB}} + \overline{\mathrm{CD}} = \overline{\mathrm{AD}} + \overline{\mathrm{CB}} = \overline{\mathrm{P}}$ when P is a vector whose direction is that of the line joining the mid-points of AC and BD.

Products of Vectors

In many cases the product of two vectors would be quite meaningless, *e.g.*, the product of two velocities. But there are many important cases in mechanics and physics as well as in geometry and algebra in which various kinds of products of two vectors are used. These involve trigonometry in the ways shown below.

Scalar Products or Dot Products

The scalar product of the vectors $\overline{OA}$, $\overline{OB}$, often denoted by $\overline{OA} \cdot \overline{OB}$ and called the dot product, means $OA \cdot OB \cos \theta$

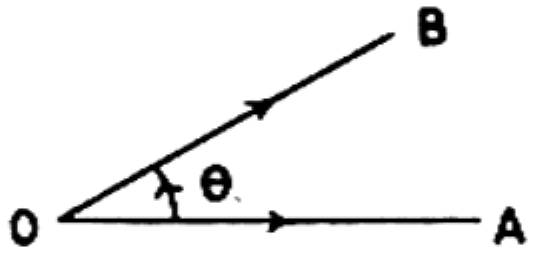

FIG. 156.

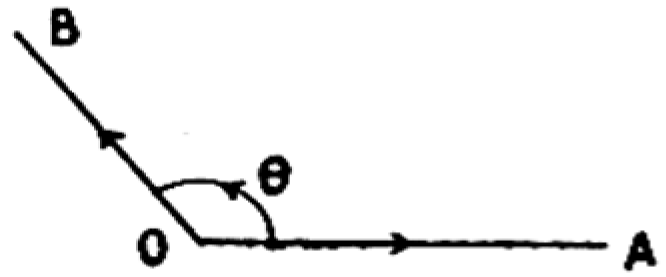

FIG. 157.

where θ is the angle between the vectors and OA, OB their lengths.

It will be seen that this is positive in Fig. 156 but negative in Fig. 157. It measures the work done by a force represented by $\overline{OA}$ while the point of application of the force moves a distance represented by $\overline{OB}$.

Vector Product or Cross Product. Areal Product

The *areal product* of the vectors $\overline{OA}$, $\overline{OB}$ means $OA \cdot OB \sin \theta$.

This measures the area of the parallelogram of which OA, OB are sides. It also measures the moment of a force $\overline{OA}$ about the point B.

If a vector $\overline{OC}$ of this magnitude is drawn at right angles to the plane OAB in the sense in which a right-handed screw, whose axis is along OC, would move if rotated from OA to OB, this vector is called the vector-product of $\overline{OA}$ and $\overline{OB}$. It is often denoted by $\overline{OA} \times \overline{OB}$ and called the cross-product as opposed to the dot product $\overline{OA} \cdot \overline{OB}$.

In the study of the cross-product or areal product you come to one of the most exciting happenings in all mathematics; see Ex. 3 below.

Both these definitions of products can be extended to the product of three vectors *which need not be in the same plane.*

In the few examples given below you are at the beginning of the subject called "*Vector Analysis*."

Examples—66

1. Calculate $\overline{OA} . \overline{OB}$ if OA $= 2$, OB $= 5$ and (i) $\theta = 40°$, (ii) $\theta = 125°$.

2. Calculate the areal products of the two vectors of Ex. 1.

3. If $\overline{OA} \times \overline{OB}$ is taken to mean either the areal product or the cross product,

prove that $\overline{OB} \times \overline{OA} = - \overline{OA} \times \overline{OB}.$

Show however that $\overline{OB} . \overline{OA} = \overline{OA} . \overline{OB}.$

(This is described by saying that the dot product is *commutative* while the cross-product is *non-commutative.* This is the first step in an algebra in which multiplication is *non-commutative.*)

4. If an areal product is thought of as measuring the moment of a force show that it is inevitable that $\overline{OB} \times \overline{OA}$ should be $- \overline{OA} \times \overline{OB}.$

5. ABCD is a rectangle with AB $= a$, BC $= b$. What are the values of $\overline{AB} . \overline{BC}$, $\overline{AB} . \overline{CD}$, $\overline{AB} . \overline{DA}$? What are the areal products corresponding to these?

6. If ρ is any vector in the plane through the origin perpendicular to a given vector α show that (i) $\rho . \alpha = 0$, (ii) the lengths of $\rho + \alpha$ and $\rho - \alpha$ are equal, (iii) the areal product of $\rho - \alpha$ and $\rho + \alpha$ is twice that of ρ and α.

7. Show that if $\overline{OA} + \overline{AB} = \overline{OB}$ then

$$\overline{OC} . \overline{OB} = \overline{OC} . \overline{OA} + \overline{OC} . \overline{AB}.$$

($\overline{OC} . \overline{OB} =$ OC . ON see Fig. 158. The right-hand side $=$ OC . OM $+$ OC . MN.)

8. If α, β, γ, δ are vectors, the result of Ex. 7 may be stated $\gamma . (\alpha + \beta) = \gamma . \alpha + \gamma . \beta$.

Show that it follows that

$$(\gamma + \delta) . (\alpha + \beta) = \gamma . \alpha + \gamma . \beta + \delta . \alpha + \delta . \beta.$$

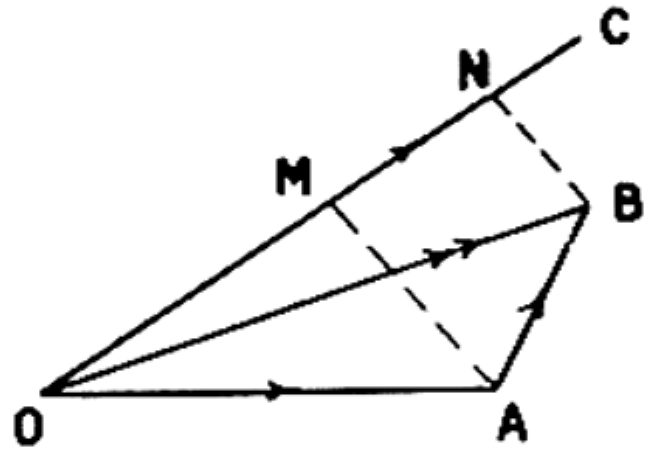

FIG. 158.

9. Criticise the following proof of Pythagoras' theorem (see Fig. 158 and suppose $\angle$OAB to be a right angle):—

$$\overline{OA} + \overline{AB} = \overline{OB}$$

$$\therefore\ (\overline{OA} + \overline{AB}) . (\overline{OA} + \overline{AB}) = \overline{OB} . \overline{OB}$$

i.e., $$\overline{OA} . \overline{OA} + \overline{2AB} . \overline{OA} + \overline{AB} . \overline{AB} = \overline{OB} . \overline{OB}$$

but $\overline{OA} . \overline{OA} = OA^2$, $\overline{AB} . \overline{AB} = AB^2$, $\overline{OB} . \overline{OB} = OB^2$, in each case since $\cos 0 = 1$, and $\overline{AB} . \overline{OA} = 0$ since $\cos 90° = 0$.

$$\therefore\ OA^2 + AB^2 = OB^2.$$

10. Show that if $\angle OAB$ is not a right angle, the same argument gives the cosine formula, explaining carefully how the negative sign is obtained.

Complete Product of Two Vectors

The dot and cross-products of "*vector analysis*" are both *partial* products.

In the subject of "*quaternions*" a *complete* product is discussed. A quaternion is the sum of a scalar and a vector. This idea—invented by Sir W. R. Hamilton over 100 years ago, long before Einstein thought of "relativity"—is thought by some to be likely to provide the proper method for dealing with that mixture of space and time which in "relativity" measures the "interval" between two "events."

In quaternions the product of two vectors is a sort of algebraic sum of the dot- and cross-products. It is defined thus :—

The product of $\overline{OA}$ and $\overline{OB} = \overline{OA} \times \overline{OB} - \overline{OA} . \overline{OB}$.

Any further discussion of this relation is beyond the scope of this book.

Product of Coplanar Vectors

If $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$ are coplanar vectors (Fig. 159), and if the triangle OBC is directly similar to the triangle ODA, then the ratio of $\overline{OC}$ to $\overline{OB}$ is defined to be the same as the ratio of $\overline{OA}$ to $\overline{OD}$.

Now $\dfrac{\overline{OC}}{\overline{OB}} = \dfrac{\overline{OA}}{\overline{OD}}$ makes the product of $\overline{OA}$, $\overline{OB}$ equal to the product of $\overline{OC}$, $\overline{OD}$. If, therefore, $\overline{OD}$ is the unit vector (along the x-axis) and may be replaced by unity, it follows that $\overline{OC}$ is the product of $\overline{OA}$, $\overline{OB}$.

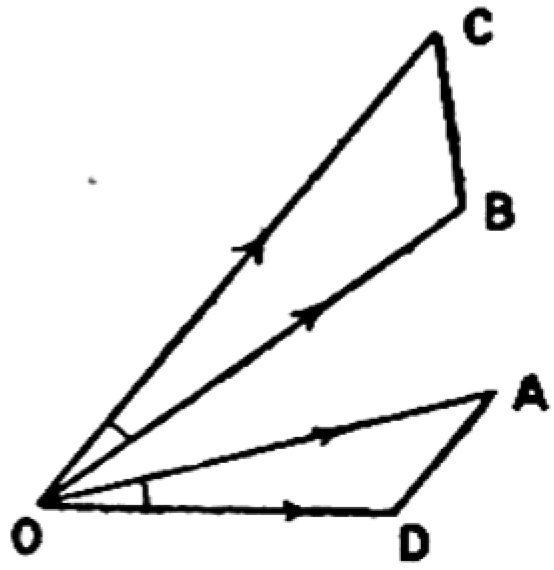

FIG. 159.

This type of product will be found to obey the ordinary laws of algebra, and so may be represented by either of the ordinary signs for multiplication (*i.e.*, as $\overline{OA} \, . \, \overline{OB}$ or as $\overline{OA} \times \overline{OB}$); it is of great importance in higher algebra and trigonometry. Unlike the dot- and cross-products it depends on the position of the axes.

Examples—67

1. If $\overline{OA}$ is of length r at an angle θ with the x-axis and $\overline{OB}$ is of length r' at an angle θ' with the x-axis, prove that $\overline{OC}$ is of length rr' and makes an angle $(\theta + \theta')$ with the x-axis. ($\overline{OD}$ is 1 along x-axis.)

Deduce that $\overline{OA} \times \overline{OB} = \overline{OB} \times \overline{OA}$, *i.e.*, that this type of multiplication is commutative.

(Note. This type of multiplication can be extended to any number of vectors, *but they must all be in the same plane.* It *cannot* be extended to three vectors which are not in the same plane.)

2. Denoting $\overline{OA}$ of Ex. 1 by (r, θ), r being its length and θ giving its direction, show that $(r, \theta)^2 = (r^2, 2\theta)$. What is the square and what the square root of $(4, 50^\circ)$?

3. From $(r, \theta) \times (r', \theta') = (rr', \theta + \theta')$ it follows that $(rr', \theta + \theta') \div (r, \theta) = (r', \theta')$.

Deduce that $(\rho, \phi) \div (r, \theta) = (\rho/r, \phi - \theta)$ and give the value of $(10, 40^\circ) \div (5, 15^\circ)$.

4. Show that if n is a positive integer,

$$(1, \theta)^n = (1, n\theta).$$

(This theorem can be extended to all values of n and is called de Moivre's theorem. It is one of the most important theorems in trigonometry.)

Ordered Pairs *

The vector OA, Fig. 160, may, if the origin and axes are given, be represented by the coordinates x and y of A. From this point of view it is an *ordered pair* of numbers (*ordered* because we must know that the number named first is x).

An ordered pair of numbers will be written with square brackets, *e.g.*, [4, 5].

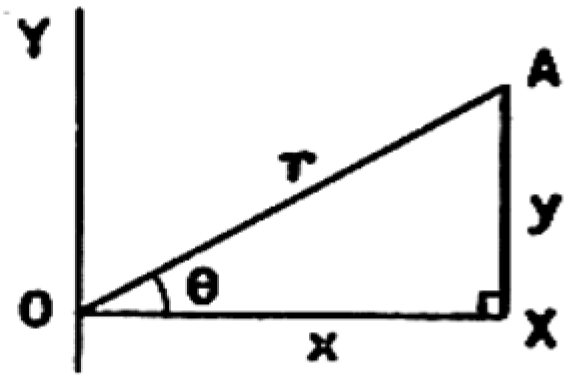

FIG. 160.

5. Show that the vector (r, θ) of Fig. 160 may be written $[r \cos \theta, r \sin \theta]$.

What vector would be denoted by $[r \sin \theta, r \cos \theta]$?

6. Show that the law for vector addition gives such results as $[4, 5] + [7, 3] = [11, 8]$ the x's and y's being added separately.

(This is another way of saying that the sum of the resolved parts of two vectors *is equal to the resolved part of their resultant.*)

7. Show that the following very curious rule for the multiplication of ordered pairs of numbers can be deduced from $(r, \theta) \times (r', \theta') = (rr', \theta + \theta')$.

$$[x_1, y_1] \times [x_2, y_2] = [x_1x_2 - y_1y_2, x_1y_2 + x_2y_1].$$

Here the $x_1x_2 - y_1y_2$ contains the $\cos \theta \cos \theta' - \sin \theta \sin \theta'$ which is equal to $\cos (\theta + \theta')$.

Use this to work out $[4, 5] \times [6, 3]$ and $[3, -2] \times [8, 5]$.

8. Show that if $[x_1, y_1] \times [x_2, y_2] = [X, Y]$, then given $[x_1, y_1]$ and $[X, Y]$ it follows that $x_2(a^2{}_1 + y^2{}_1) = Xx_1 + Yy_1$ and deduce a similar formula for y_2.

As $[X, Y] \div [x_1, y_1] = [x_2, y_2]$, these results are formulæ for division.

9. Use the method or the result of Ex. 9 to find the values of :

(i) $[2, 23] \div [4, 5]$. (ii) $[46, -18] \div [6, 2]$.

---

* An *ordered pair* of numbers is often called a *complex number* and is written $x + iy$ where i stands for $\sqrt{(-1)}$. What meaning can be attached to this apparently nonsensical expression is left for you to learn in books on higher trigonometry or algebra.

TRIGONOMETRY ON THE SPHERE

The Lune

Any two great circles cross each other at the ends of a diameter, *i.e.*, at opposite "poles." The surface of the sphere enclosed by two great semicircles is called a *lune*.

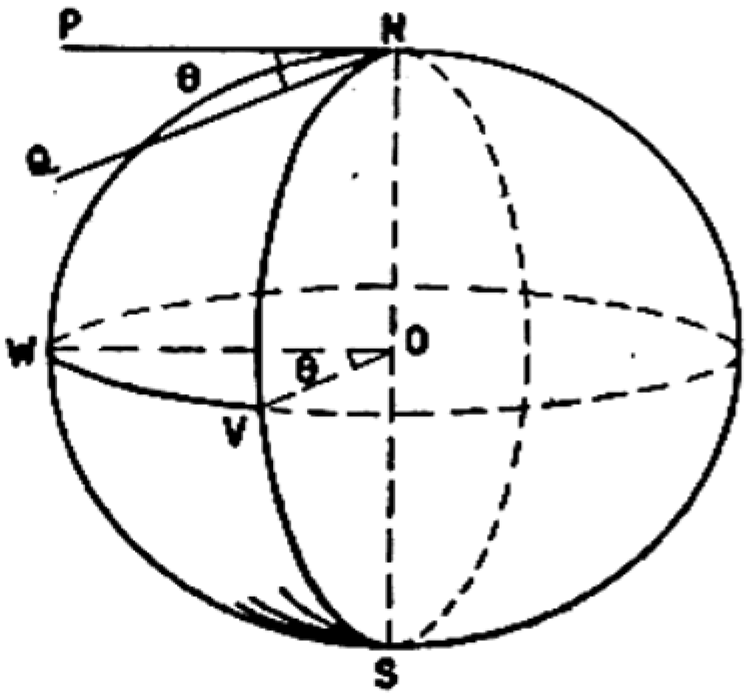

FIG. 161.

In Fig. 161 the meridians of longitude NWS, NVS form a lune. The angle of the lune is the angle between the planes NWS, NVS, and is the angle WOV (where WV is the equator and O the centre) or the angle PNQ between the tangents at N to the two semicircles.

The *area* of the lune whose angle is θ° is θ/360 of the area of the sphere, *i.e.*, $\frac{\theta}{360} \times 4\pi R^2$. If the angle is x radians then the area is $\frac{x}{2\pi} \times 4\pi R^2$ or $2xR^2$.

Spherical Triangles

If a third great circle cuts the two already drawn, the lune is divided into two spherical triangles, figures formed by the arcs of three great circles. The equator in Fig. 161 divides the lune into two congruent isosceles triangles NWV, SWV, whose angles are θ, 90°, 90°.

If the third great circle cuts the two first circles obliquely, as in Fig. 162, eight spherical triangles are formed, for the great circles CB and CA make four lunes, and great circle BA divides each lune into two triangles. Each triangle is the "image" of the triangle formed by its antipodes: for instance, △A′B′C′ is the image of △ABC (they are related like the left and right hands).

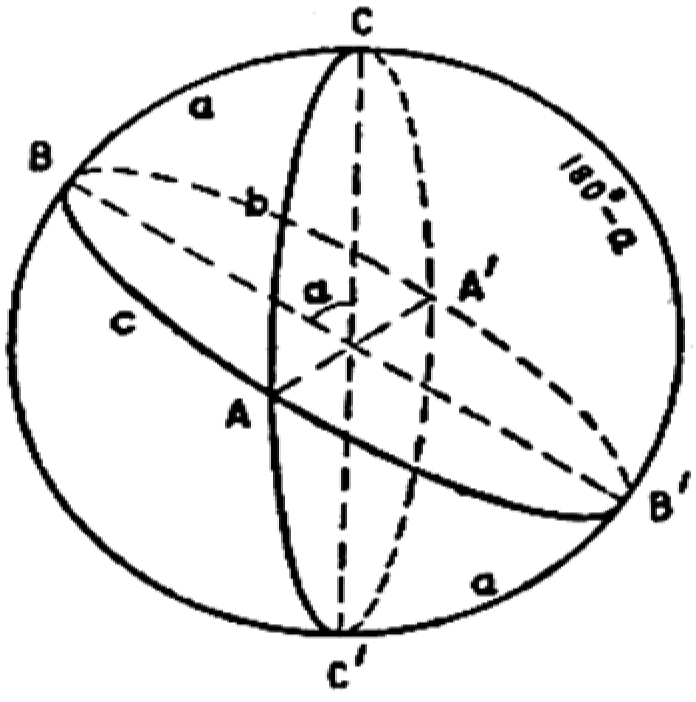

FIG. 162.

The Sides and Angles of a Spherical Triangle

The angles A, B, C of the triangle ABC are the angles between the tangents to the sides at the points A, B, C. The sides BC, CA, AB are denoted by a, b, c as in the case of plane triangles, but here the letters stand for the angles subtended by the sides at the centre. If a, b, c are in radian measure, the *actual* lengths of BC, CA and AB are aR, bR and cR. When the Earth is the sphere it is easy to convert the angle a into a distance since 1° on a great circle represents 60 *nautical* miles, *i.e.*, 1 minute of angle to the nautical mile.

If the sides a, b, c are in degrees, the eight spherical triangles of Fig. 162 consist of four pairs with the following sides :—

a, b, c ($\triangle$ABC and $\triangle$A′B′C′) ; $180° - a$, $180° - b$, c ($\triangle$ABC′ and $\triangle$A′B′C).
a, $180° - b$, $180° - c$ ($\triangle$AB′C′ and $\triangle$A′BC), $180° - a$, b, $180° - c$ ($\triangle$AB′C and $\triangle$A′BC′).

The Area of a Spherical Triangle

If the area of $\triangle$ABC is denoted by $\triangle$, and A, B, C are the angles in *radians*, then from Fig. 162,

area of $\triangle$ABC′ $= 2\text{C} \cdot \text{R}^2 - \triangle$ (since lune CBC′A $= 2\text{C} \cdot \text{R}^2$)
,, ,, $\triangle$AB′C $= 2\text{B} \cdot \text{R}^2 - \triangle$ (,, ,, BCB′A $= 2\text{B} \cdot \text{R}^2$)
,, ,, $\triangle$AB′C′ $= 2\text{A} \cdot \text{R}^2 - \triangle$ (,, ,, AC′A′B $= 2\text{A} \cdot \text{R}^2$)
Also $\triangle$ABC $= \triangle$, and, by addition,

Area of front hemisphere

$$= 2\text{A} \cdot \text{R}^2 + 2\text{B} \cdot \text{R}^2 + 2\text{C} \cdot \text{R}^2 + \triangle - 3\triangle$$

$$\therefore\ 2\pi\text{R}^2 = 2(\text{A} + \text{B} + \text{C})\text{R}^2 - 2\triangle$$

$$\therefore\ \triangle = (A + B + C - \pi)R^2 = ER^2.$$

Hence the area of a spherical triangle depends on the excess E radians of its angle sum over π radians or 180°. $\text{E} = \text{A} + \text{B} + \text{C} - \pi$ (the *spherical excess*). It follows that two triangles of different sizes cannot have the same angles, or they would have equal areas ! Similar triangles of different sizes on a sphere do not exist. The smaller the triangle the nearer to 180° is its angle-sum.

Spherical Trigonometry

Formulæ, resembling those of plane trigonometry, can be found which connect the "sides" a, b, c and the angles A, B, C of a spherical triangle. These formulæ are needed in navigation since the shortest sailing or flying route between two points is along the arc of a great circle. This forms a spherical triangle with the meridians of longitude through the two points. In Fig. 162 the shortest route from A to B is along the great circle arc AB.

The Sine Formula

In Fig. 163, P is the projection of A on the plane BOC, O being the centre of the sphere. AM and AN are perpendicular to OB, OC and so are "lines of greatest slope" of the planes AOB, AOC if BOC is thought of as horizontal.

$$\therefore\ \angle AMP = B\ (\angle \text{ between planes AOB, BOC})$$
$$\angle ANP = C\ (\angle\ \ \text{,,}\ \ \text{,,}\ \ \text{AOC, BOC}).$$

Now $AM \sin B = AP = AN \sin C$

But $\angle AMO = 90° = \angle ANO$ (constructed) and

$$\angle AOM = c,\ \angle AON = b.$$
$$\therefore\ AM = R \sin c,\ AN = R \sin b$$
$$\therefore\ R \sin c \,.\, \sin B = R \sin b \,.\, \sin C.$$
$$\therefore\ \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

and similarly each $= \dfrac{\sin a}{\sin A}$.

$$\therefore\ \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

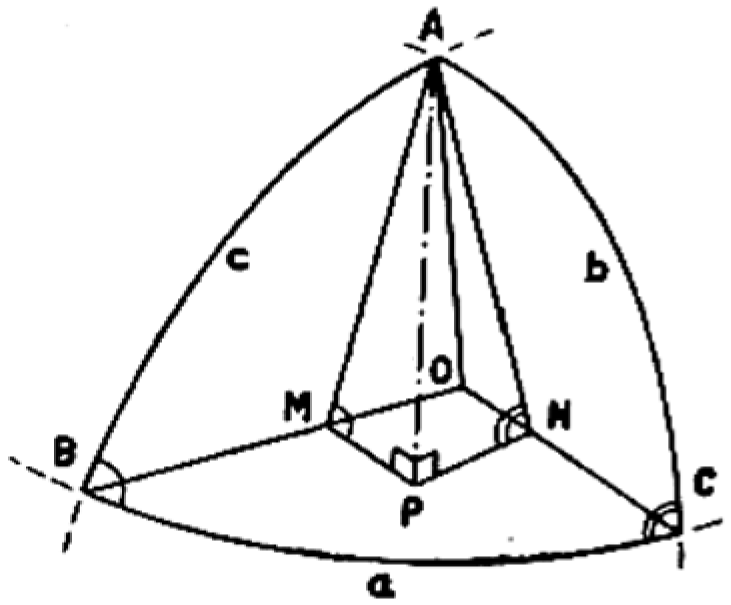

FIG. 163.

It is easy to see that if P falls outside $\angle BOC$ the proof still holds good.

N.B. In a spherical triangle C is *not* equal to $180° - (A + B)$, etc.

The Cosine Formula

In Fig. 164 AT, AU are the tangents at A to the arcs AB, AC, meeting OB, OC at T, U. $\angle TAO = 90° = \angle UAO$.

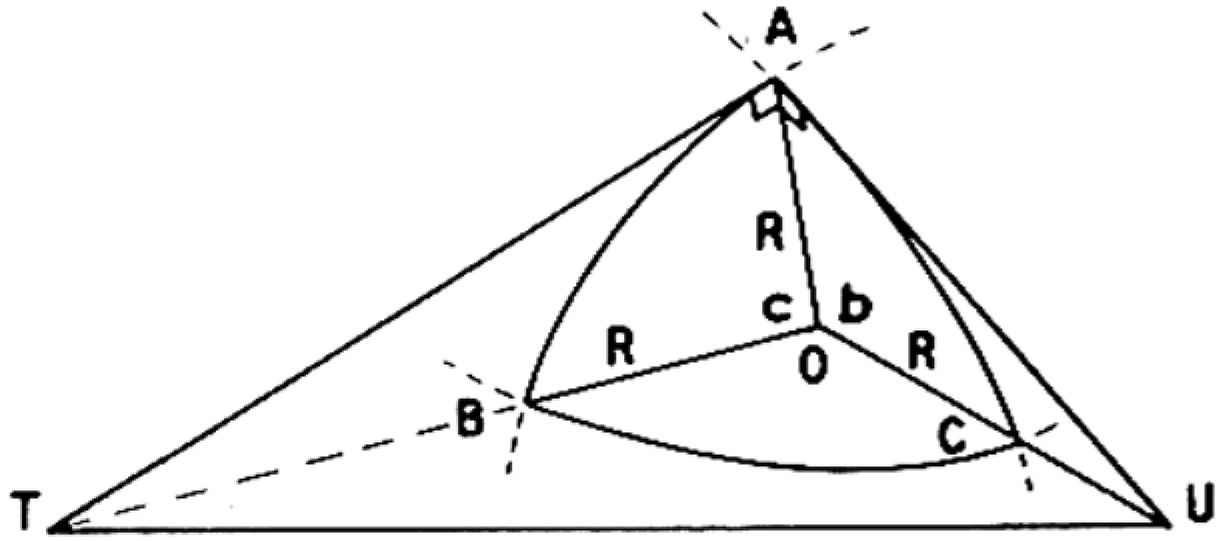

FIG. 164.

In $\triangle OTU$, $OT = R \sec c$, $OU = R \sec b$

In $\angle ATU$, $AT = R \tan c$, $AU = R \tan b$.

Applying the cosine formula to these two triangles, since $\angle TOU = a$, $\angle TAU = A$,

$$TU^2 = R^2 \sec^2 c + R^2 \sec^2 b - 2R^2 \sec c \,.\, \sec b \,.\, \cos a.$$
$$TU^2 = R^2 \tan^2 c + R^2 \tan^2 b - 2R^2 \tan c \,.\, \tan b \,.\, \cos A.$$

Subtracting, dividing by R^2, and recalling that

$$\sec^2 c - \tan^2 c = 1 = \sec^2 b - \tan^2 b,$$
$$0 = 1 + 1 - 2 \sec c \,.\, \sec b \,.\, \cos a + 2 \tan c \,.\, \tan b \,.\, \cos A.$$

Rearranging, dividing by 2 and multiplying by $\cos c \,.\, \cos b$,

$\mathbf{\cos a = \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, \cos A}$, and, similarly,
$\cos b = \cos c \,.\, \cos a + \sin c \,.\, \sin a \,.\, \cos B$,
$\cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C$.

In this proof angles b and c are acute, but the formula can be shown to hold good in all cases.

Examples—68

1. Taking the Earth's radius as 4000 miles, what is the area of the lune between the meridian of Greenwich and the meridian 15° W. ?

2. What is the area of the spherical triangle formed by the equator and the two meridians of No. 1 ? Check by finding the spherical excess in radians and using the formula $\triangle = ER^2$.

3. If in Fig. 162 the angles A, B, C are $\frac{1}{4}\pi$, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, find (i) the angles and (ii) the areas of the triangles A′BC, ABC′, AB′C, taking the radius as R.

4. If A′, B′, C′ are three points on a great circle equally spaced at 120° apart and A, B, C are three points near A′, B′, C′ on the same side of the great circle, then as A, B, C get closer and closer to A′, B′, C′ respectively, ABC becomes the maximum possible spherical triangle ; to what value do its angles tend and what is its spherical excess, which is the greatest excess possible for a spherical triangle.

5. If the angles of a spherical triangle are measured in *degrees*, and D is the spherical excess in *degrees*, what is the area of a spherical triangle on the Earth's surface whose excess is D ? Take R = 4000 miles.

6. An equilateral triangle on the Earth's surface has sides 200 miles long. Find its approximate area by assuming it to be plane. Using the result of No. 5, find the spherical excess of the spherical triangle and hence the size of each angle. [In surveying, allowance is made for the curvature of the Earth in very accurate work. The result of this question shows the sort of allowance necessary.]

7. Use the cosine formula to find the side c of a spherical triangle in which $a = \frac{1}{2}\pi$, $b = \frac{1}{3}\pi$, $C = \frac{1}{4}\pi$.

8. Find the distance, along a great circle track, from Liverpool (53½°N, 3°W) to Baroda (29°N, 73°E).

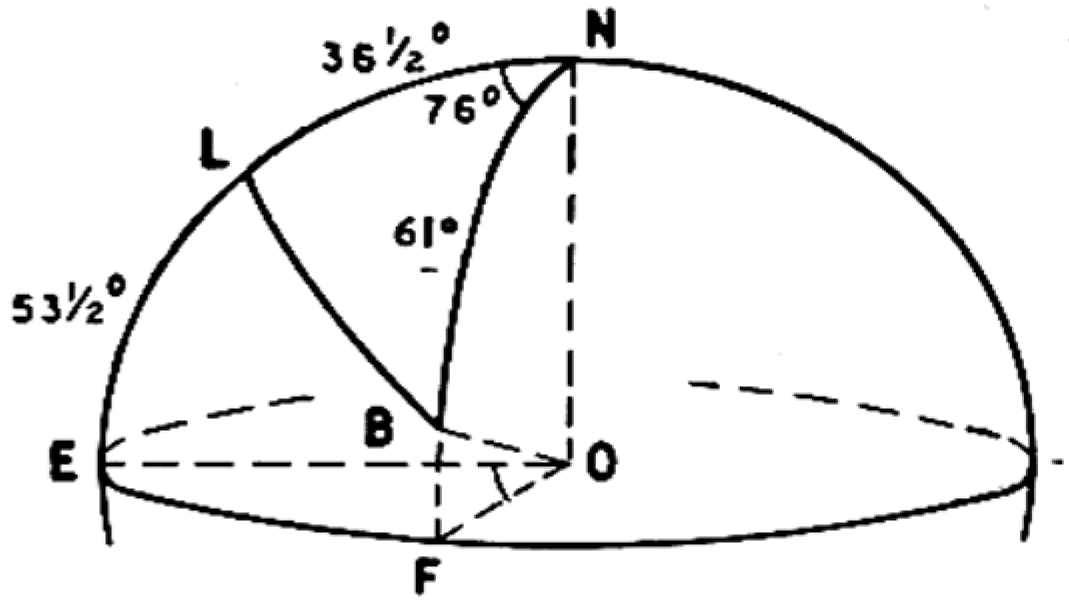

FIG. 165.

[Sol'n. The triangle formed by the North Pole, London and Baroda is used.

$$\text{LN} = 36\tfrac{1}{2}^\circ \ (\angle\text{LOE} = 53\tfrac{1}{2}^\circ, \text{ latitude})$$
$$\text{NB} = 61^\circ \ (\angle\text{BOF} = 61^\circ, \text{ latitude})$$
$$\angle\text{LNB} = 76^\circ \ (3^\circ\text{W to } 73^\circ\text{E, difference of longitude}).$$

In the triangle LNB we know l, b, N and require n.

$$\begin{aligned}\cos n &= \cos l \cos b + \sin l \sin b \cos \text{N} \\ &= \cos 61^\circ \cos 36\tfrac{1}{2}^\circ + \sin 61^\circ \sin 36\tfrac{1}{2}^\circ \cos 76^\circ.\end{aligned}$$

From this, n is found. If this angle is expressed in minutes the distance LB is obtained in nautical miles, in which form the answer should be given.]

9. Deduce the formula

$$\cos A = (\cos a - \cos b \,.\, \cos c)/\sin b \sin c,$$

and write down the expressions for cos B and cos C. Find the angles of a triangle whose sides are $a = 60^\circ$, $b = 90^\circ$, $c = 120^\circ$.

10. Find the great-circle distance from London (51½°N, 0°) to New York (41°N, 74°W) in nautical miles.

11. A and B are in latitude 60° and their longitudes differ by 60°. Find their distance apart (i) along the great circle, (ii) along the parallel of latitude. Repeat with the difference of longitude 120°. Use nautical miles.

12. Reference to a globe will show that the compass-direction changes along a great-circle track. Find the direction of the track in No. 9 (i) on leaving Liverpool (this will be $\angle$NLB), (ii) on approaching Baroda (this will be obtained from $\angle$NBL). [Hint: use the sine formula on $\triangle$LNB, the value of n having been found. There will be two possible values for L since sin L is found, and two for B though it is fairly clear from a rough figure that B is acute. We can use $\cos L = (\cos l - \cos b \cos n)/\sin b \sin n$ to decide whether L is acute or obtuse, for, if $\cos l < \cos b \cos n$, cos L is negative and L obtuse, since l, b, n are acute.]

13. Find the distance from Aberdeen (57°N, 2°W) to Sydney (34°S, 152°E) and the bearings of the start and finish of the journey.

The "Four Parts" Formula

The cosine formula gives the third side when two sides and the included angle are known, or any angle when the three sides are known. If two angles are known the third is *not* obtainable from an angle sum, as in a plane triangle, and this restricts the usefulness of the sine formula. Another formula is needed and will now be obtained.

Since $\cos a = \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, \cos A$

and $\cos b = \cos c \,.\, \cos a + \sin c \,.\, \sin a \,.\, \cos B$, we get by substitution for $\cos b$ in the first formula

$$\cos a = (\cos c \,.\, \cos a + \sin c \,.\, \sin a \,.\cos B) \cos c + \sin b \,.\, \sin c \,.\, \cos A.$$

$$\therefore \cos a\,(1 - \cos^2 c)$$
$$= \sin c \,.\, \sin a \,.\, \cos c \,.\, \cos B + \sin b \,.\, \sin c \,.\, \cos A.$$

Dividing by $\sin c \,.\, \sin a$ (since $1 - \cos^2 c = \sin^2 c$)

$$\frac{\cos a}{\sin a} \,.\, \sin c = \cos c \,.\, \cos B + \frac{\sin b}{\sin a} \,.\, \cos A \quad \text{but} \quad \frac{\sin b}{\sin a} = \frac{\sin B}{\sin A}.$$

$$\therefore \mathbf{\cot a \,.\, \sin c = \cos c \,.\, \cos B + \sin B \,.\, \cot A.}$$

This formula connects *four consecutive parts* a, B, c, A of a spherical triangle. It is unsymmetrical and hard to remember, but the following scheme is helpful :—

(i) Write down the *cotangents* of the *sides* in order, followed by those of the *angles* in the same order.

$$\cot a \qquad \cot c \qquad \cot B \qquad \cot A.$$

(ii) Express the two middle cotangents as cos/sin,

$$\cot a \quad \frac{\cos c}{\sin c} \quad \frac{\cos B}{\sin B} \quad \cot A.$$

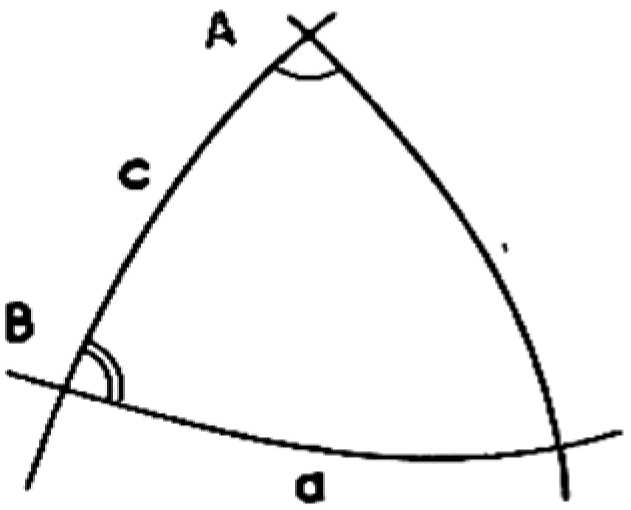

FIG. 166.

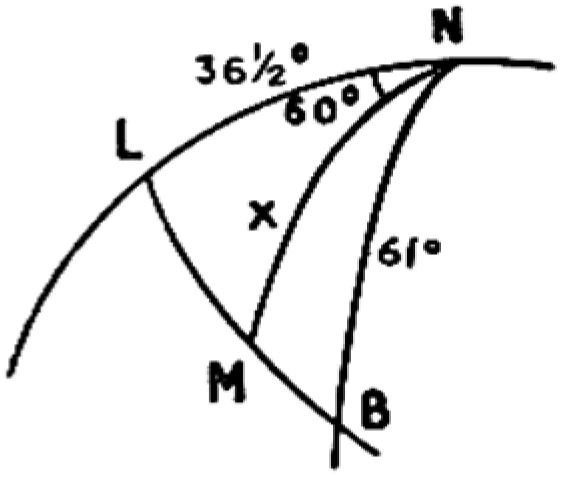

FIG. 167.

(iii) Form the products indicated by the dotted lines

$$\cot a \sin c = \cos c \cos B + \sin B \cot A.$$

The two first steps are easily done mentally.

The "Four Parts" formula is useful in finding intermediate points on a great circle track. The disadvantage of this shortest track is that its direction changes continuously, making it difficult to steer by compass. The rhumb course has a constant compass bearing and is used on shorter journeys where the saving by following the shortest track is negligible.

Worked Example

In No. 9 of the previous set, find where the track crosses the meridian of longitude 60°E.

From No. 12 the angle L is found to be 82°, or it can be found from the "Four Parts" formula. For in $\triangle$LNB the parts are 61°, 36½°; 76°, L.

$$\therefore \cot 61^\circ \sin 36\tfrac{1}{2}^\circ = \cos 36\tfrac{1}{2}^\circ \cos 76^\circ + \sin 76^\circ \cot L$$

$$\therefore \cot L = \cot 61^\circ \sin 36\tfrac{1}{2}^\circ \operatorname{cosec} 76^\circ - \cos 36\tfrac{1}{2}^\circ \cot 76^\circ.$$

Now from $\triangle$LNM the four parts are x, 36½°; 60°, 82°.

$$\therefore \cot x \,.\, \sin 36\tfrac{1}{2}^\circ = \cos 36\tfrac{1}{2}^\circ \,.\, \cos 60^\circ + \sin 60^\circ \cot 82^\circ$$

$$\therefore \cot x = \cot 36\tfrac{1}{2}^\circ, \cos 60^\circ + \sin 60^\circ \cot 82^\circ, \operatorname{cosec} 36\tfrac{1}{2}^\circ$$

$$\therefore x \bumpeq 48^\circ 36'$$

∴ the latitude of M is 41½°N approx.

The Polar Triangle

The distance of any point on the Earth's equator from the North (or South) Pole is 90° measured along a meridian. For any point P on a sphere there is a great circle whose relation to the point is that of Pole and equator ; it is called the *polar circle* of P.

Similarly, for any great circle there are two *poles*. In Fig. 168, $\triangle$A′B′C′ is the *polar triangle* of $\triangle$ABC. The point A′ is that pole of the great circle through BC which is on the same side of BC as A ; similarly for B′ and C′.

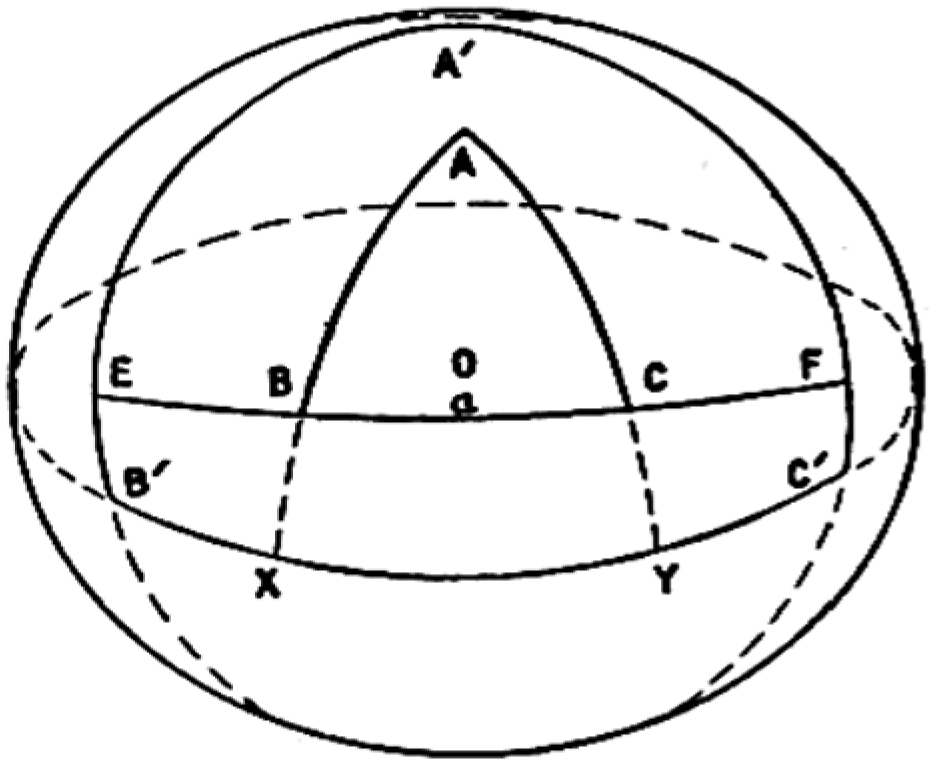

FIG. 168.

The polar triangle has interesting properties.

1. The original $\triangle$ABC is the polar triangle of $\triangle$A′B′C′.

For B′ is a pole of AC, $\therefore$ B′ is 90° from A
And C′ ,, ,, ,, ,, AB, $\therefore$ C′ ,, ,, ,, A
$\therefore$ A is a pole of B′C′, and it is on the same side as A′.
Similarly B, C are poles of C′A′, A′B′.

2. The sides a', b', c' of the polar triangle are $\pi - \text{A}$, $\pi - \text{B}$, $\pi - \text{C}$.

If the great circle BC produced cuts A′B′, A′C′ at E, F, then the arc EF measures the angle A′, since the angle between the " meridians " A′B′, A′C′ is the " difference of longitude " along the " equator " EBCF.

$$\therefore \text{A}' = \text{EF} = \text{EB} + \text{BC} + \text{CF}.$$
$$= (\tfrac{1}{2}\pi - a) + a + (\tfrac{1}{2}\pi - a) \text{ since}$$
$$\text{EC} = \tfrac{1}{2}\pi = \text{BF}.$$
$$\therefore \text{A}' = \pi - a \text{ and similarly } \text{B}' = \pi - b,\ \text{C}' = \pi - c.$$

3. The angles A′, B′, C′ of the polar triangle are $\pi - a$, $\pi - b$, $\pi - c$.

If AB, AC meet B′C′ at X, Y, then arc XY = A. For XY is the arc of the " equator " B′C′ cut off by the " meridians " AX, AY.

$$\therefore a' = \text{B}'\text{C}' = \text{B}'\text{X} + \text{XY} + \text{YC}'$$
$$= (\tfrac{1}{2}\pi - \text{A}) + \text{A} + (\tfrac{1}{2}\pi - \text{A}) \text{ since } \text{B}'\text{Y} = \tfrac{1}{2}\pi = \text{C}'\text{X}.$$
$$\therefore a' = \pi - \text{A}, \text{ and similarly } b' = \pi - \text{B},\ c' = \pi - \text{C}.$$

4. The formulæ already obtained (the Sine, Cosine, and "Four Parts") apply equally to the polar triangle. For the cosine formula,

$$\cos a' = \cos b' \cos c' + \sin b' \,.\, \sin c' \cos A'$$

and substituting for a', b', c', A' their values in terms of A, B, C, a,

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - c)\cos(\pi - a),$$

i.e., $-\cos A = (-\cos B.)(-\cos C) + \sin B.\ \sin C.(-\cos a).$
or $\cos a \sin B \sin C = \cos A + \cos B \cos C,$

a useful formula enabling a side to be found when the three angles are known. It is called, for obvious reasons, the supplement of the cosine formula.

5. If any of the formulæ for a spherical triangle are expressed in terms of the supplements of the angles, instead of the angles themselves, so that, for instance, the cosine formula reads $\cos a = \cos b \cos c - \sin b \sin c \cos(\pi - A)$. Then, by considering the Polar Triangle we see that *the sides and the angle-supplements may be exchanged*,
$\therefore \cos(\pi - A)$
$= \cos(\pi - B)\cos(\pi - C) - \sin(\pi - B)\sin(\pi - C)\cos a.$
This is the formula just given in No. 4.

Examples—69

1. Show that the sine formula is its own supplement.

2. Use the supplement of the cosine formula to find b if (i) $A = B = C = 70°$; (ii) $A = 40°$, $B = 65°$, $C = 100°$.

3. Obtain the supplements of the "Four Parts" formulæ which connect (i) a, B, c, A; (ii) A, b, C, a. Are the new formulæ different in type from the old ?

Solution of Spherical Triangles

4. a, b, c given. Find A, B, C if $a = 45°$, $b = 90°$, $c = 120°$ [Cosine formula.]

5. A, B, C given. Find the sides if $A = 60°$, $B = 120°$, $C = 90°$. [Supplement of cosine formula.]

6. a, b, c given. Find c, A, B if $a = 120°$, $b = 135°$, $c = 30°$.
[c from cosine formula ; A, B from sine formula.]

7. A, B, c given. Find C, a, b if $A = 45°$, $B = 60°$, $c = 90°$.

[C from supplement of cosine formula; a, b from sine formula.]

8. In such a flight as that from London ($51\frac{1}{2}$°N, 0°) to Vancouver ($49\frac{1}{2}$°N, 123°E) it is important to know how near the North Pole the track goes.

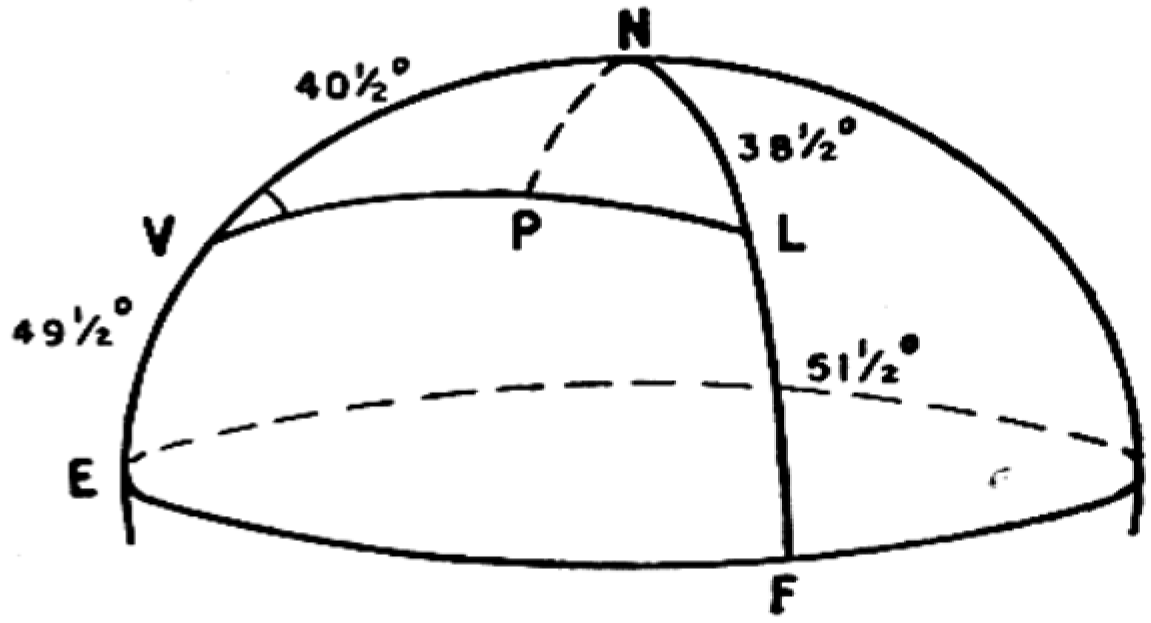

FIG. 169.

[In △NVL we first find angle V. The "four parts" are $88\frac{1}{2}°$, $40\frac{1}{2}°$; $123°$, V, and the corresponding formula
$\cot 38\frac{1}{2}° \,.\, \sin 40\frac{1}{2}° = \cos 40\frac{1}{2}° \cos 123° + \sin 123° \cot V$.
Hence V is obtained.

If NP is a great circle arc from N to a point P on the track, then by the sine formula

$$\frac{\sin NP}{\sin V} = \frac{\sin 40\frac{1}{2}°}{\sin NPV} \quad \therefore \sin NP = \frac{\sin 40\frac{1}{2}° \sin V}{\sin NPV}.$$

But sin V is known, so sin NP is least when sin NPV is greatest, *i.e.*, when NPV = 90°. NP can now be found, and the highest latitude reached in the flight will be known.]

9. Solve the same problem for a flight from London to New York (41°N, 74°W).

10. By substituting $C = 90°$ in the appropriate sine, cosine, supplement of cosine, and four-parts formulæ prove the following for a *right-angled* spherical triangle :—

(i) $\sin a = \sin c \sin A$; (ii) $\sin b = \sin c \sin B$;
(iii) $\cos c = \cos a \cos b$; (iv) $\cos c = \cot A \cot B$;
(v) $\cos A = \sin B \cos a$; (vi) $\cos B = \sin A \cos b$;
(vii) $\tan a = \tan A \sin b$; (viii) $\tan b = \tan B \sin a$;
(ix) $\tan a = \tan c \cos B$; (x) $\tan b = \tan c \cos A$.

11. Write down the supplements of the above formulæ. These will be applicable to a triangle in which $c = 90°$; a *quadrantal* triangle.

12. Find the longitude of the point E where the great circle track from Aberdeen to Sydney crosses the equator. (See No. 13 of Examples 66.)

[Hint: Draw the meridian-arc NE. In $\triangle$ANE, NE$=90°$; find $\angle$ANE using the value of $\angle$NAS already obtained.]

CHAPTER IX

TOPICS OF TRIGONOMETRICAL INTEREST

1. Projectiles

Fig. 170 shows the path of a cricket ball thrown with a speed of 80 ft. per second at an angle of 30° with the horizontal, if the resistance of the air is ignored.

If there were no " gravity," the ball would take the straight path OT, and after t secs. OT $= 80t$ ft. But gravity causes any heavy object to fall $16t^2$ ft. in t secs., so the ball is actually at P, $16t^2$ ft. below T.

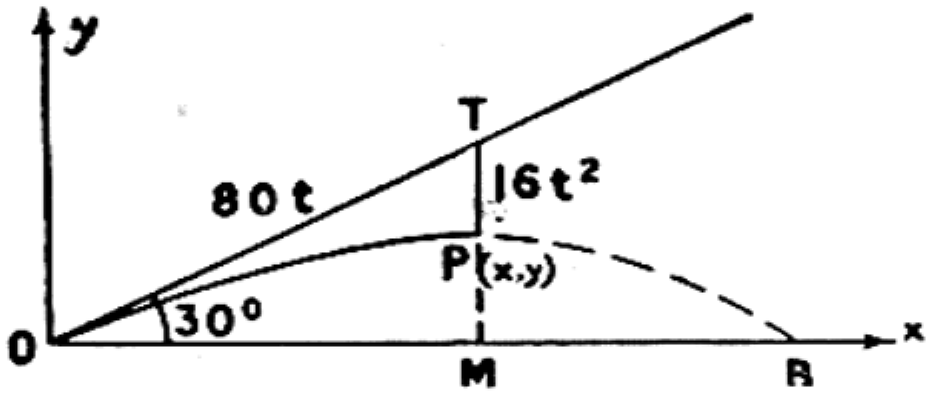

Fig. 170.

1. What are the coordinates of T after 1, 2, $2\frac{1}{2}$ secs. ?

2. ,, ,, ,, ,, ,, P ,, ,, ,,

3. If the ball strikes the ground at B, what is the range OB ?

4. Assuming the highest point of the path to be reached at half-time, what is the greatest height ?

5. If angle BOT is α instead of 30°, what are the coordinates of (i) T, (ii) P after t secs. ?

6. After what time is P on the ground ? [Find when $y = 0$.]

7. Show that the range is $200 \sin 2\alpha$.
[*N.B.* $\sin 2\alpha = 2 \sin \alpha \cos \alpha$.]

8. (i) For what value of α is this range greatest ? (ii) What is the maximum range ?

9. What is the greatest height reached when the range is a maximum ? [Find the value of y at half-time.]

10. Find the range and greatest height reached by a shell fired at 4000 ft. per second at an angle of 45°.

2. SEEING ENGLAND

England can be regarded as a triangle BLD, whose corners are at Berwick-on-Tweed, Lands End and Dover. BL = 425 mi., LD = 325 mi., DB = 350 mi. approximately.

1. Calculate one of the angles of this triangle, and so find the radius R of the circle through BLD. [$b = 2\text{R} \sin \text{B}$.]

2. From a sufficient height above the centre of this circle it would be possible to see the whole of England. The distance in miles of the horizon visible from a height of h feet is almost exactly $\sqrt{1\frac{1}{2}h}$. Find the height for which the distance of the visible horizon is equal to R.

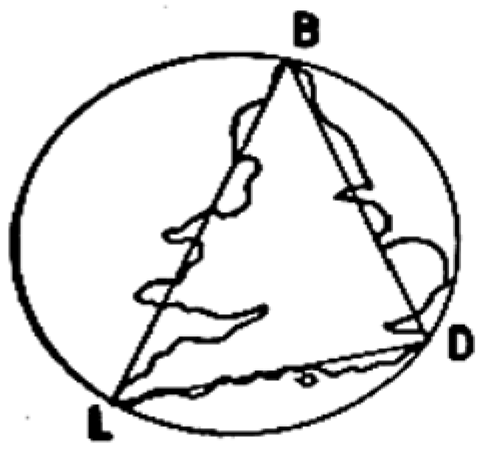

FIG. 171.

3. Locate the centre of the circle approximately from a map. This gives the least height from which the whole of England could be seen.

4. What is the dip of the horizon from the aircraft ?

3. SOLUTION OF EQUATIONS BY SKETCH GRAPHS AND TABLES

For equations such as those of B, p. 137, for which a graphic solution is necessary, it is often easier to use sketch graphs and tables.

Consider the equation $\sin \theta = \frac{1}{2}\theta$ of Ex. 8, p. 138 (θ being the radian measure).

Draw a rough sketch of $y = \sin \theta$ from $\theta = 0$ to π and the line $y = \frac{1}{2}\theta$, got by joining $(0, 0)$ to $(\pi, 1{\cdot}57)$. This shows at once that the line cuts upwards across the curve between 90° and 120° ($\theta = \frac{1}{2}\pi$ and $\frac{2}{3}\pi$).

Then form the tables below ; (i) proves the angle between 100° and 110°, considerably nearer to 110°, (ii) proves this angle between 108° and 109°, the distance from these being roughly as 3 : 2 ; (iii) proves that to the nearest minute the angle is 108°36′ and θ is 1·8954 with an error less than ·0002.

The work as far as (ii) is sufficient for such accuracy as an accurate graph would give.

(i)

$\theta\angle$	100°	110°	120°
$\sin\theta$	·9848	·9397	·8660
θ	1·5708	1·5708	1·5708
	·1745	·3491	·5236
θ	1·7453	1·9199	2·0944
$\frac{1}{2}\theta$	·8726	·9599	1·0472

(ii)

$\theta\angle$	106°	108°	109°
$\sin\theta$	·9613	·9511	·9454
θ	1·5708	1·5708	1·5708
	·2793	·3142	·3316
θ	1·8501	1·8840	1·9024
$\frac{1}{2}\theta$	·9250	·9420	·9512
	diff.	·0091	·0057

(iii)

$\theta\angle$	108° 40′	35′	36′
$\sin\theta$	·9474	·9479	·9478
θ	1·5708	1·5708	1·5708
	·3258	·3243	·3246
θ	1·8966	1·8951	1·8954
$\frac{1}{2}\theta$	·9483	·9475	·9477

Note that 1·5708 is for $\frac{1}{2}\pi$. As θ is above this, its value has to be obtained by the additions shown.

1. Solve the equation $\cos x^\circ = x/90$, finding the angle either (i) to the nearest degree, or (ii) to the nearest tenth of a degree.

2. Repeat for the equation $1 - \sin x = \frac{2}{3}x$ where x is in radians. The examples B. 1 to 6 of p. 137 are suitable for this method.

4. THE LADDER PROBLEM

A ladder 30 ft. long leans against a warehouse wall. A packing case 7 ft. high and 5 ft. wide just touches the ladder and the wall. How high above the ground is the top of the ladder?

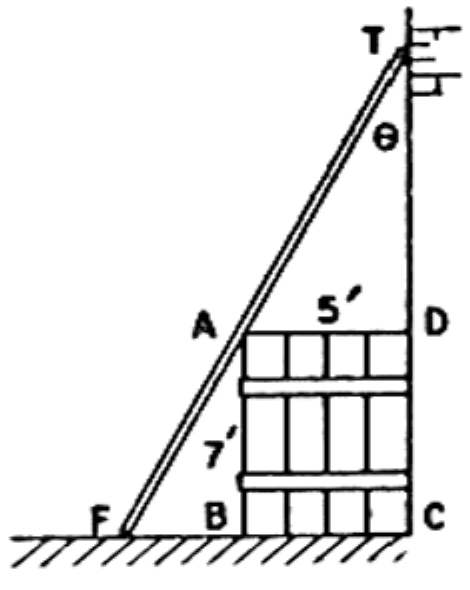

FIG. 172.

1. Suppose the ladder makes an angle θ with the vertical. Express AT and AF in terms of θ.

2. Deduce the equation $7\sec\theta + 5\operatorname{cosec}\theta = 30$.

3. Draw the graph of $7\sec\theta + 5\operatorname{cosec}\theta$ from 0° to 90° and hence solve the above equation.

4. Express the height TC in terms of θ and calculate its value from the results of No. 3.

If this problem is attempted by algebra it leads to an equation of the fourth degree.

5. THE DONKEY PROBLEM

A donkey is tethered by a rope to a post at the edge of a circular meadow of radius 1 chain. How long is the rope if the donkey can graze exactly half the meadow ?

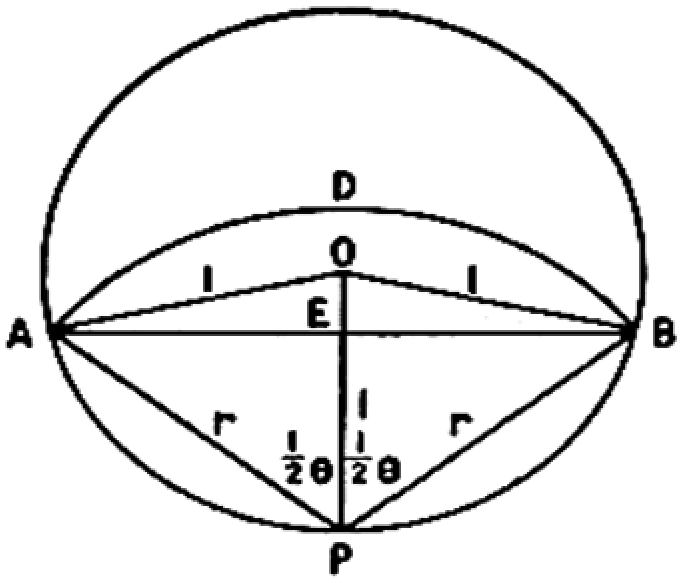

FIG. 173.

If r is the length of the rope and $\angle APB = \theta$, prove

(i) $\angle AOB = 2\pi - 2\theta$.

(ii) $r = 2\cos\frac{1}{2}\theta$.

(iii) Area sector OAPB $= (\pi - \theta)$.

(iv) Area segment APB $= (\pi - \theta + \sin\theta\cos\theta)$.

(v) Area sector PADB $= \frac{1}{2}r^2\theta$.

(vi) ,, segment ADB $= \frac{1}{2}r^2(\theta - \sin\theta)$.

(vii) $\frac{1}{2}r^2 = 1 + \cos\theta$.

(viii) Area grazed $= \pi - \sin\theta + \theta\cos\theta$ and this $= \frac{1}{2}\pi$ (half the area of the ⊙).

(ix) It should be clear that O must be between D and E. Find the value of θ when (*a*) D and O coincide, (*b*) E and O coincide.

(x) Graph the expression $\sin\theta - \theta\cos\theta$ between the (radian) values of the angles obtained in (ix), (*a*) and (*b*), and hence solve the equation $\sin\theta - \theta\cos\theta = \frac{1}{2}\pi$.

(xi) Evaluate r, from (ii), to the nearest six inches.

[*N.B.* (iv). The area of $\triangle$AOB is $-\sin\theta\cos\theta$ since θ is obtuse.]

6. THE PYTHAGORAS' FIGURE

This is not strictly the figure for Pythagoras' theorem as ABC is any triangle whatever, but BCDE, CAFG, ABHK are squares.

Prove : 1. The areas of $\triangle$AFK, $\triangle$BEH, $\triangle$CDG and $\triangle$ABC are equal. [Use $\frac{1}{2}bc\sin A$.]

2. $KF^2 + EH^2 + DG^2 = 3(a^2 + b^2 + c^2)$.

3. $AE^2 + BG^2 + CK^2 = 2(a^2 + b^2 + c^2) + 12\triangle$.

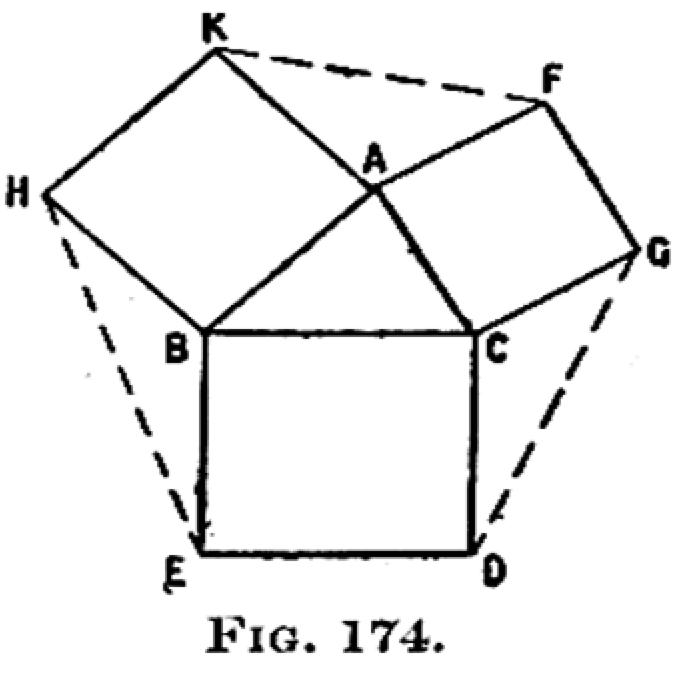

FIG. 174.

7. QUADRILATERALS

1. Use Fig. 175 to show that the area of any quadrilateral ABCD is $\frac{1}{2}xy \sin \theta$, where x, y are the lengths of its diagonals and θ is the angle between them. [Show that the area ABCD is half that of the parallelogram : similarly arrowed lines are parallel.]

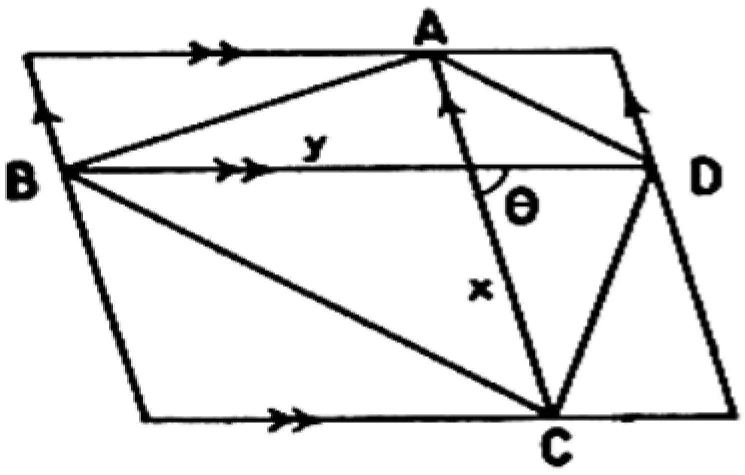

FIG. 175.

2. The sides AB, BC, CD, DA and the diagonal AC of a quadrilateral are 12, 11, 10, 9 and 8 inches long. Calculate the length of the other diagonal using formulæ suitable for use with logarithm tables.

3. The quadrilateral ABCD, whose sides AB, BC, CD, DA are 11, 12, 8 and 6 inches, is cyclic. Calculate (i) its angles, (ii) its diagonals, (iii) its area, (iv) the angle between the diagonals.

[Use the cosine formula to find two expressions for AC^2, remembering that $\cos D = -\cos B$.]

8. THE ORTHOCENTRE AND PEDAL TRIANGLE

The three altitudes AD, BE, CF of a triangle ABC meet at a point usually denoted by H and called the *orthocentre*. The triangle DEF is called the *pedal* triangle, since DEF are the *feet* of the altitudes.

[In Ex. 1 to 9 the triangle ABC is acute-angled.]

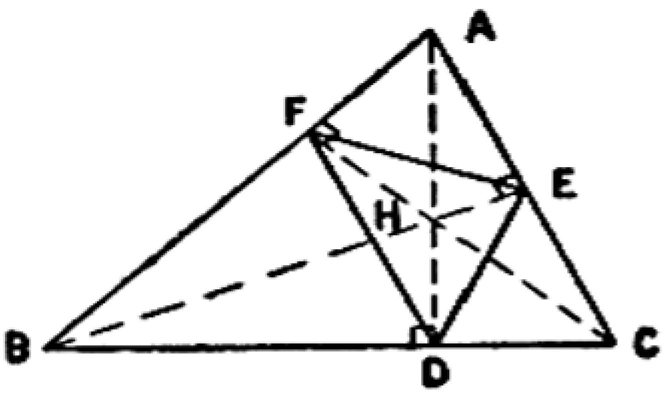

FIG. 176.

1. Prove that $AD = 2R \sin B \sin C$.
2. Prove $\angle AHE = C$, and $AH = AE \operatorname{cosec} C = 2R \cos A$. What are BH and CH ?
3. Prove $DH = 2R \cos B \cos C$. [$\angle CHD = B$.]

4. From AD = AH + HD we get

$$\sin B \sin C = \cos A + \cos B \cos C.$$

Explain this.

5. Prove EF $= a \cos A$. What are FD, DE? [$\triangle$AEF is similar to $\triangle$ABC.]

6. Show that the angles of $\triangle$DEF are $180° - 2A$, $180° - 2B$, $180° - 2C$.

7. Show that the radius of the circumcircle of $\triangle$DEF is $\frac{1}{2}R$. [Use $a = 2R \sin A$.]

8. Show that the area of $\triangle$DEF is $\triangle \cos A \cos B \cos C$.

9. What is the orthocentre of $\triangle$BHC?

10. If A is obtuse, prove that the angles of its pedal triangle are $2A - 180°$, $2B$, $2C$. What are its sides? [Remember that cos A is negative.]

9. Some Geometrical Theorems

1. Two triangles are similar, with angles A, B, C. Their corresponding sides are a, b, c; na, nb, nc. Show that their areas are in the ratio $1 : n^2$. [Use $\frac{1}{2}bc \sin A$.]

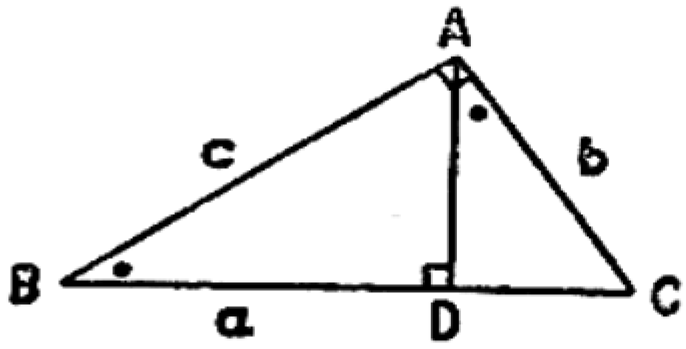

Fig. 177.

2. The triangle ABC has a right angle at A, and D is the projection of A on BC. Show that the three triangles in the figure (177) are similar.

3. What sides in $\triangle$ABD and $\triangle$ACD correspond to a in $\triangle$ABC? What is the ratio of the areas of triangles ABC, ABD, ACD? If $\triangle$ABC has area Δ, what are the areas of $\triangle$ABD and $\triangle$ACD? Hence prove Pythagoras' theorem.

4. Show that the cosine formula is equivalent to the two Extensions of Pythagoras' theorem.

5. Prove Apollonius' theorem, that, if AM is a median of $\triangle$ABC, $b^2 + c^2 = 2AM^2 + \frac{1}{2}a^2$. [Apply the cosine formula to $\triangle$s AMB, AMC.]

10. The Angle-bisector of a Triangle

AX bisects ∠A and meets BC at X. AX $= f$.

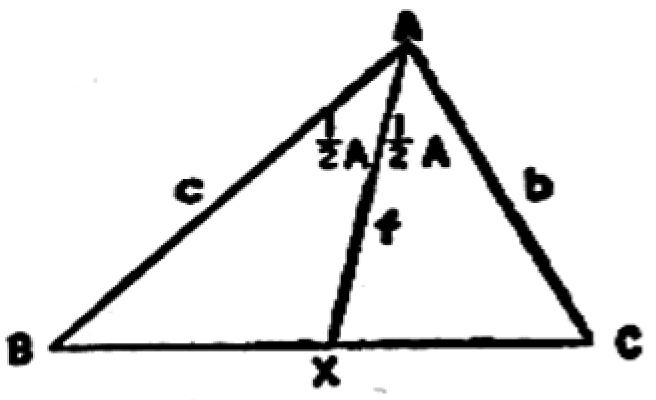

Fig. 178.

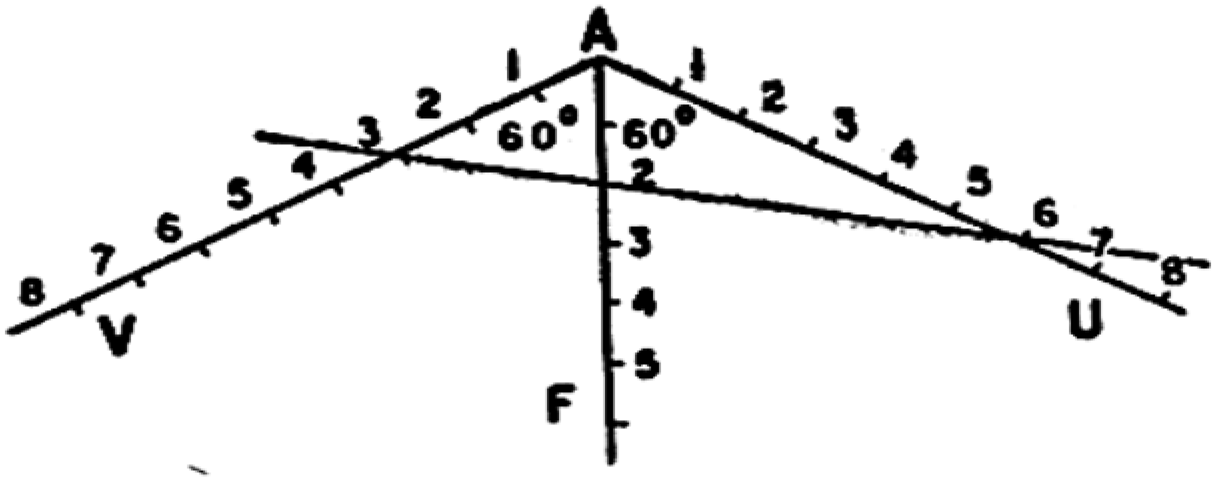

Fig. 179.

1. Prove BX : XC $= c : b$, using the sine formula. [Show that BX $= c \sin \frac{1}{2}$A/sin AXB, etc.]

2. Prove $f = 2bc \cos \frac{1}{2}\mathrm{A}/(b + c)$. [$\Delta = \frac{1}{2}bc \sin \mathrm{A} = \triangle\mathrm{ABX} + \triangle\mathrm{ACX} = bc \sin \frac{1}{2}\mathrm{A} \cos \frac{1}{2}\mathrm{A}$.]

3. If A = 120°, prove $\frac{1}{f} = \frac{1}{b} + \frac{1}{c}$. This relation can be used to make a *nomogram* for use with optical calculations involving the formula $\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$. Draw three lines AV, AF, AC with ∠VAF = ∠FAU = 60°. Graduate all three equally. A ruler laid along $v = 3$, $u = 6$ will cross AF at $f = 2$, since $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

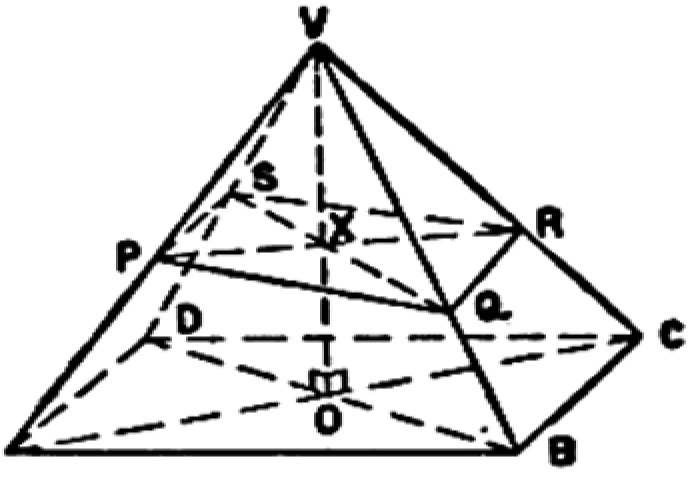

Fig. 180.

4. A pyramid on a square base ABCD and with vertex V is cut by an oblique plane PQRS (Fig. 180). Use the fact that VX is the bisector of both the angles SVQ and PVR to prove that 1/VP + 1/VR = 1/VS + 1/VQ.

11. TRIANGULATION

When a large area of country is to be mapped a *base-line* AB, perhaps several miles long, is first measured with extraordinary care and accuracy.

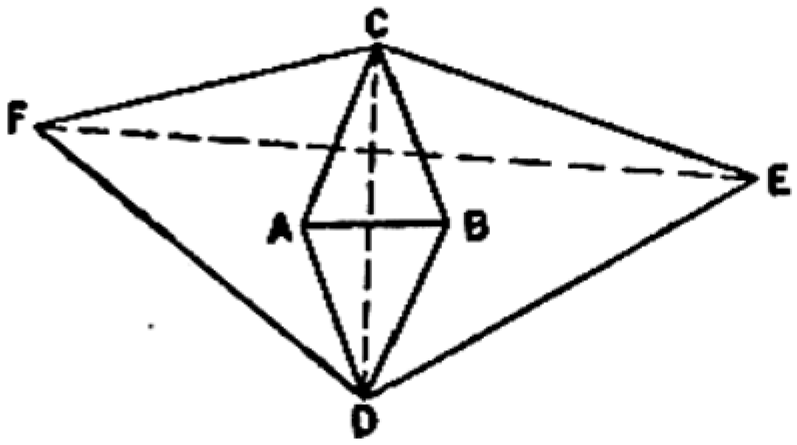

FIG. 181.

The position of C and D, two prominent landmarks, are fixed by measurement of the angles BAC, ABC, BAD, ABD. From C and D further landmarks E, F can be fixed, and so on.

AB is 5000 yards, $\angle BAC = 65°43'$, $\angle ABC = 76°54'$, $\angle BAD = 75°31'$, $\angle ABD = 64°20'$, $\angle CDE = 87°54'$, $\angle DCE = 73°10'$, $\angle CDF = 66°44'$, $\angle DCF = 84°20'$.

1. Solve the triangles ABC, ABD.
2. Solve the triangles ACD, BCD. This gives CD in two ways, providing an important check.
3. Using CD as a new base-line, solve triangles CDE, CDF.
4. Solve triangles CEF, DEF to obtain the third base line EF.

It is assumed that all the points concerned are at the same level.

NOTE. The base line used for the survey of England and Wales was laid out on Salisbury Plain. It is eight miles long and took nearly six months to measure.

12. THE REGULAR TETRAHEDRON

A regular tetrahedron is a triangular pyramid whose faces are equilateral triangles. If the edges are all of length 2a, find

1. The height.
2. The volume.
3. The angle between a slant edge and the base.
4. The angle between a face and the base.
5. The distance between the mid-points of opposite edges, such as VA, BC.

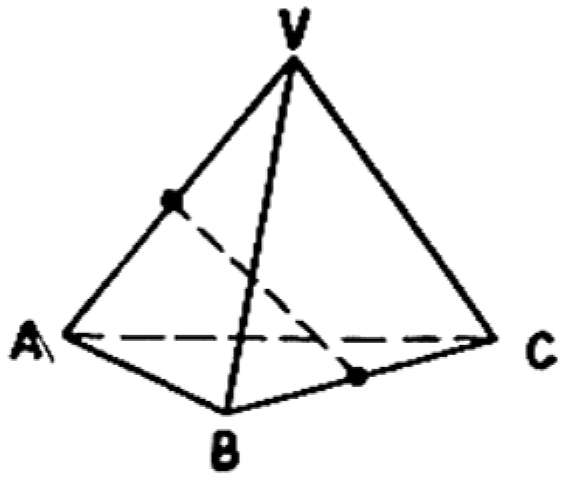

FIG. 182.

6. The radius of the sphere through V, A, B, C.
7. The radius of the sphere touching the six *edges*.
8. The radius of the sphere touching the four *faces*.

13. PTOLEMY'S THEOREM

The first trigonometrical tables worthy of the name were compiled by the Greek Ptolemy, who lived about A.D. 150. He discovered a geometrical fact which enabled him to prove what we call the addition theorems.

Ptolemy's theorem states that if ABCD is a cyclic quadrilateral, and AC,BD are the diagonals,

$$AC \,.\, BD = AB \,.\, CD + AD \,.\, BC,$$

or, in words, 'the product of the diagonals is equal to the sum of the products of the pairs of opposite sides.'

1. What is the length of a chord which subtends $\angle A$ at the circumference of a circle of radius R ?

2. In Fig. 183 ABCD is a cyclic quadrilateral in which AC is a diameter of the circle, whose radius is R. If $\angle CAD = \alpha$, $\angle BAC = \beta$, express the sides and diagonals of ABCD in terms of R, α, β, and show that Ptolemy's theorem gives

$$\sin(\alpha + \beta) = \sin\alpha \,.\, \cos\beta + \cos\alpha \,.\, \sin\beta.$$

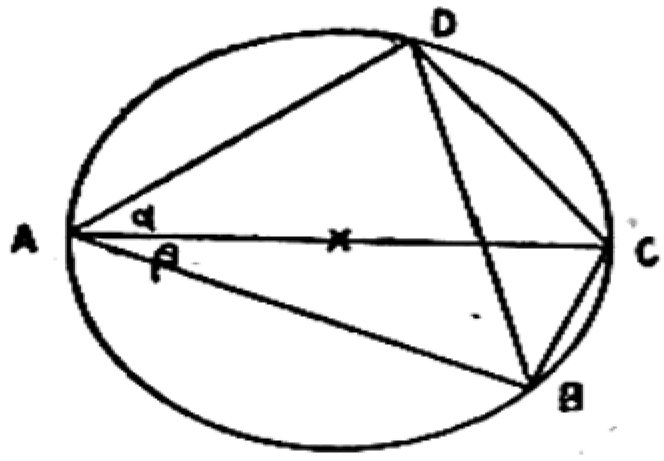

FIG. 183.

3. Imagine $\triangle ABC$ folded about AC so that B comes between D and C on the circle. Apply 'Ptolemy' to the new cyclic quadrilateral ACBD to prove

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \,.\, \sin\beta.$$

4. Deduce formulæ for $\cos(\alpha \pm \beta)$ by writing $(90 - \alpha)$ instead of α.

5. What theorem is obtained by applying 'Ptolemy' to a rectangle ?

6. Take ABCD to be four consecutive corners of a regular pentagon of unit side. Let x be the length of the diagonals, which are equal. Apply 'Ptolemy' and solve the resulting quadratic equation, leaving the solution (only one applies) in square-root form. Hence prove that $\cos 36° = \frac{1}{4}(\sqrt{5} + 1)$. Deduce $\sin 18°$, $\cos 18°$, $\sin 36°$ as decimals.

[Hipparchus, about 150 B.C., compiled a table of chords at intervals of 1°. Ptolemy's tables, about 300 years later, reduced the interval to 6′.]

14. TRIGONOMETRY IN PHYSICS

1. When a beam of light AG strikes a glass surface part is reflected along GB and part refracted along GC into the glass. NGL is at right angles to the surface. Sin $\alpha = \mu \sin \beta$. Prove that, when $\angle BGC = 90°$, $\tan \alpha = \mu$. [α is then called the 'polarising angle.'] (Fig. 184.)

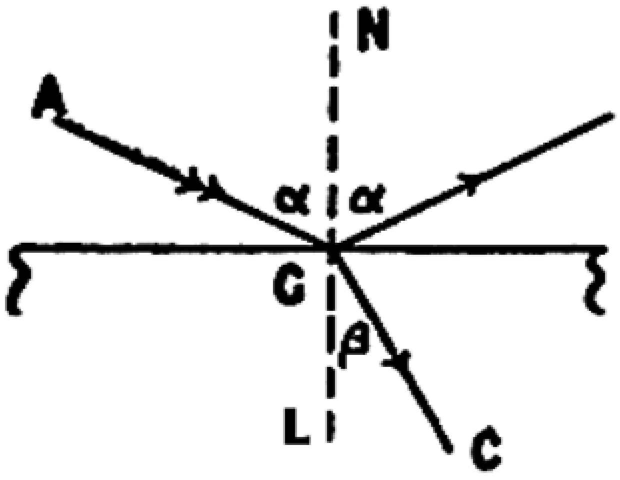

FIG. 184.

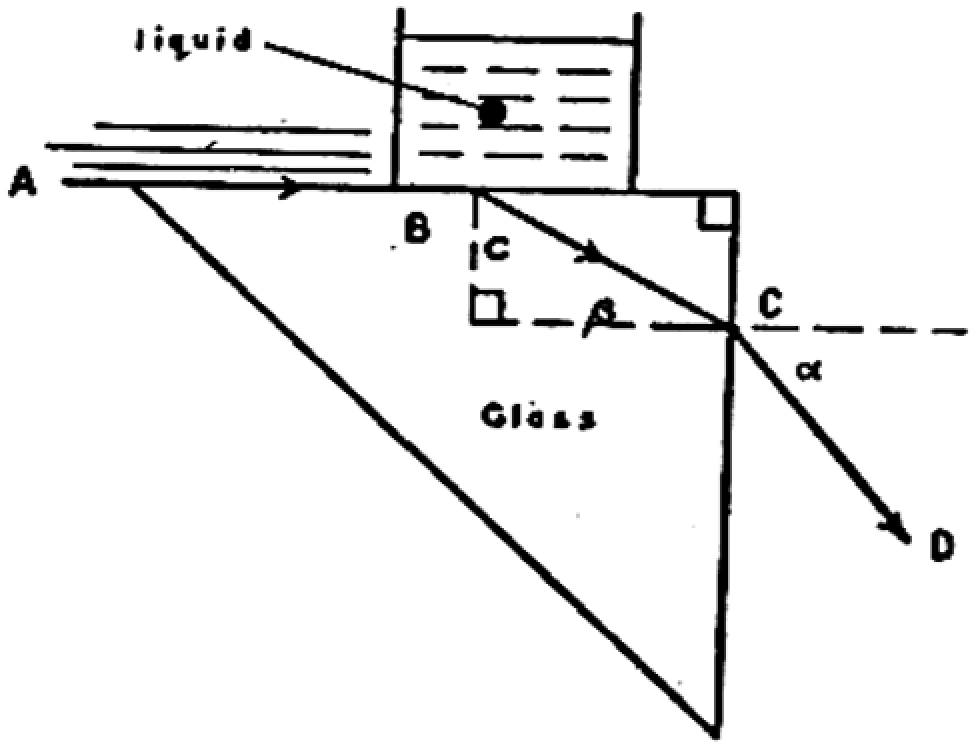

FIG. 185.

2. In Fig. 185 a ray of light takes the path ABCD. Sin $\alpha = \mu \sin \beta$ and $\sin c = l/\mu$. Express l in terms of μ and α. [Pulfrich Refractometer.]

3. The formula $\mu = \sin\frac{1}{2}(D + A)/\sin \frac{1}{2}A$ is used in spectrometry. If A is very small, prove that $D = (\mu - 1)A$. If $\mu = 1\frac{1}{2}$, $A = 10°$, find the percentage error in using the second formula to obtain D.

4. In Fig. 186, if α and β are small, prove that $\alpha : \beta = e : f$ [Theory of Telescopes.]

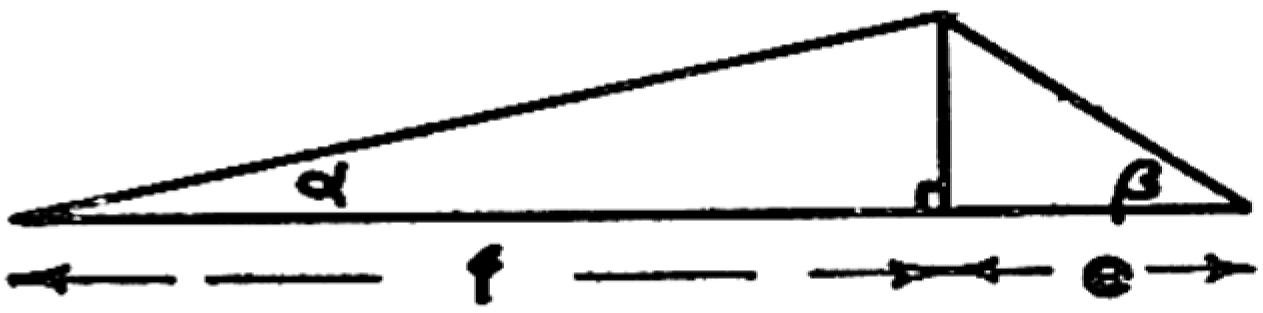

FIG. 186.

5. An object B at the bottom of a pond appears to be at A to an eye at E. ECA is a straight line, SAB and DC are perpendicular to the water-surface SC, and $\sin \alpha = \mu \sin \beta$.

Show that, if α is small, the true depth SB is μ times the apparent depth SA. [If α is small, $\sin\alpha \bumpeq \alpha \bumpeq \tan\alpha$.] (Fig. 187.)

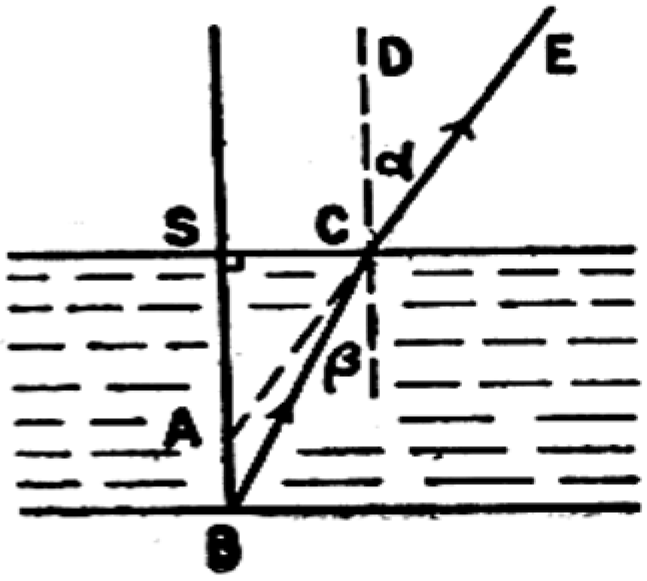

FIG. 187.

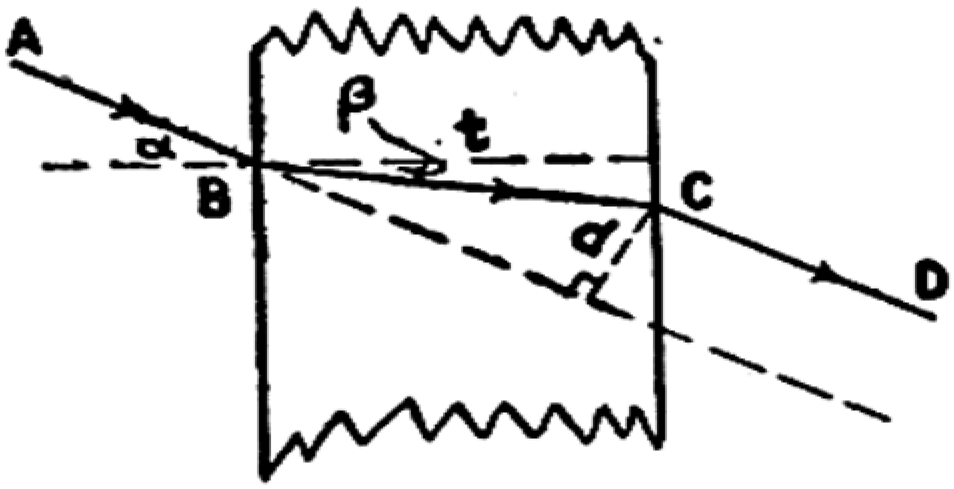

FIG. 188.

6. ABCD is a ray of light passing through a glass plate of thickness t. AB||CD and $\sin\alpha = \mu\sin\beta$. Find a formula for the sideways shift d of the ray, in terms of t and α, and show that if α is small, $d = t\left(1 - \frac{1}{\mu}\right)\alpha$. (Fig. 188.)

7. UHV is a ray of light refracted on entering a cylinder of glass of radius r. UG $= u$, VG $= v$ and arc HG $= a$.

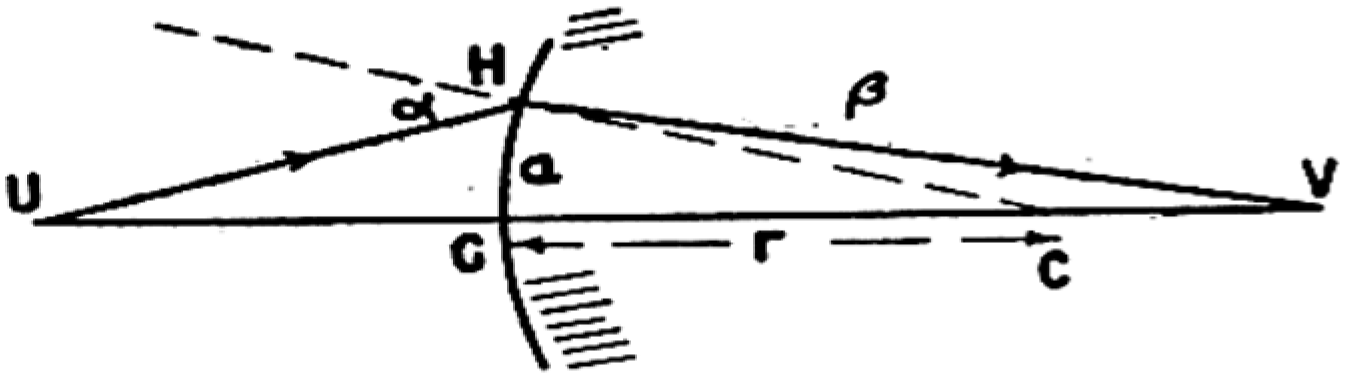

FIG. 189.

If all the angles are small, and $\sin\alpha = \mu\sin\beta$ (*i.e.*, $\alpha \bumpeq \mu\beta$), prove $\frac{\mu}{v} + \frac{1}{u} = \frac{\mu - 1}{r}$.

[$U = a/u$, $\alpha = U + a/r$, $C = V + \beta$.]

15. Construction of Regular Polygons

Of regular polygons with 12 sides or less it is not possible to construct accurately those with 7, 9 or 11 sides. It is fairly easy to construct polygons with 5 or 10 sides, but the method is usually quickly forgotten.

If a protractor is allowed, then 5-, 9- and 10-gons are easy enough.

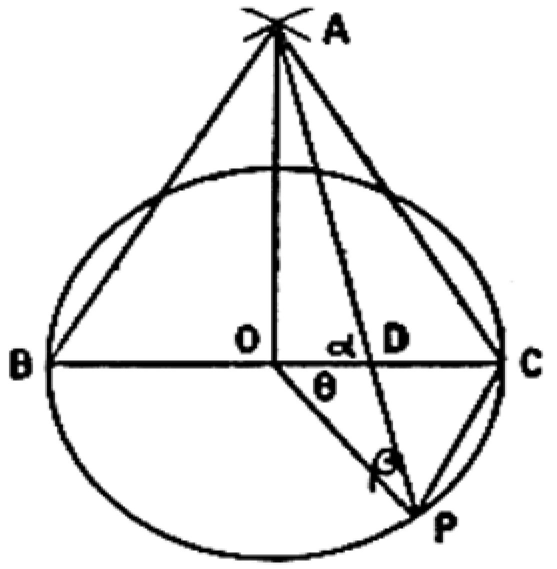

FIG. 190.

De Ville's method, illustrated in Fig. 190, is used by draughtsmen and happens to be exact for 3, 4 and 6 and a close approximation for 5, 7 or more sides.

It is supposed that we wish to inscribe a regular n-sided figure in a given circle, centre O and diameter BC. ABC is an equilateral triangle and DC is $2/n$ of BC.

If AD meets the circle at P, then CP is a side of the n-gon, and the figure is completed by 'stepping' round the circle.

Taking BC as $2n$ units in each case,

1. Prove that if $n = 6$, $\angle COP = 60°$, without using tables. [Calculate OA ($= 6\sqrt{3}$ or $\sqrt{108}$), AD, $\sin \alpha$, $\cos \alpha$, $\sin \beta$ (from sine formula) $\cos \beta$, $\sin \theta$ ($= \sin \overline{\alpha - \beta}$).]
2. Calculate θ when $n = 5$, and the percentage error. [Percentage error = actual error/true length, expressed as a percentage.]
3. In a regular heptagon (7 sides) $\angle COP$ would be 360°/7. Calculate the percentage error in the angle obtained by the above method.
4. Repeat No. 3 for a regular nonagon (9 sides).

16. THE CONSTRUCTION OF A SUNDIAL

To us, the earth seems fixed and the sun appears to move in a large circular arc at a steady rate of 15° per hour.

At the north pole, a vertical stick would cast a shadow which turned steadily at 15° per hour on horizontal ground.

In a lower latitude λ (Fig. 191), if GL is a stick parallel to the Earth's axis, and a flat surface ML, parallel to the equator was dug out, the shadow of GL on this flat surface would rotate at 15° per hour. But we should prefer the shadow to be cast on a horizontal (or vertical) surface.

In Fig. 192 the shadow of GL on a horizontal surface at L is DL. At mid-day the shadow is HL. $\angle GLH = \angle LHM = \lambda$, the latitude, as is clearly seen in Fig. 191. In order to graduate the sundial we need to know the relation between ϕ (the angle on the dial), θ and λ. This is simply the hillside problem again.

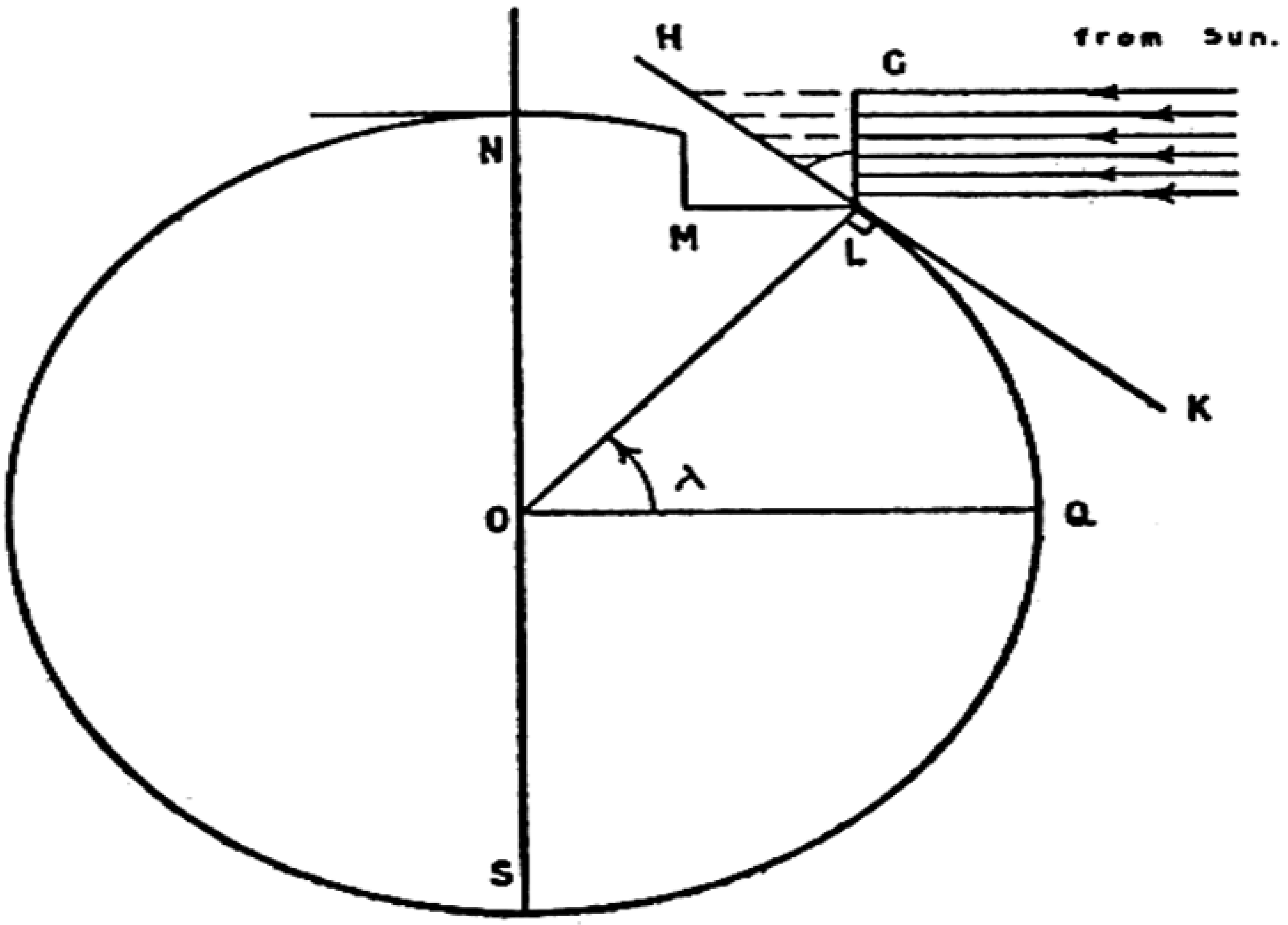

FIG. 191.

1. Prove $\tan \phi = \sin \lambda \,.\, \tan \theta$.

2. Calculate the values of ϕ for the latitude of London ($51\frac{1}{2}°$) when θ has the values 15°, 30°, 45°, 60°, 75°, 90°, 105°, which correspond to 1, 2, 3, 4, 5, 6 and 7 p.m. (or 11, 10, 9, 8, 7, 6 and 5 a.m.).

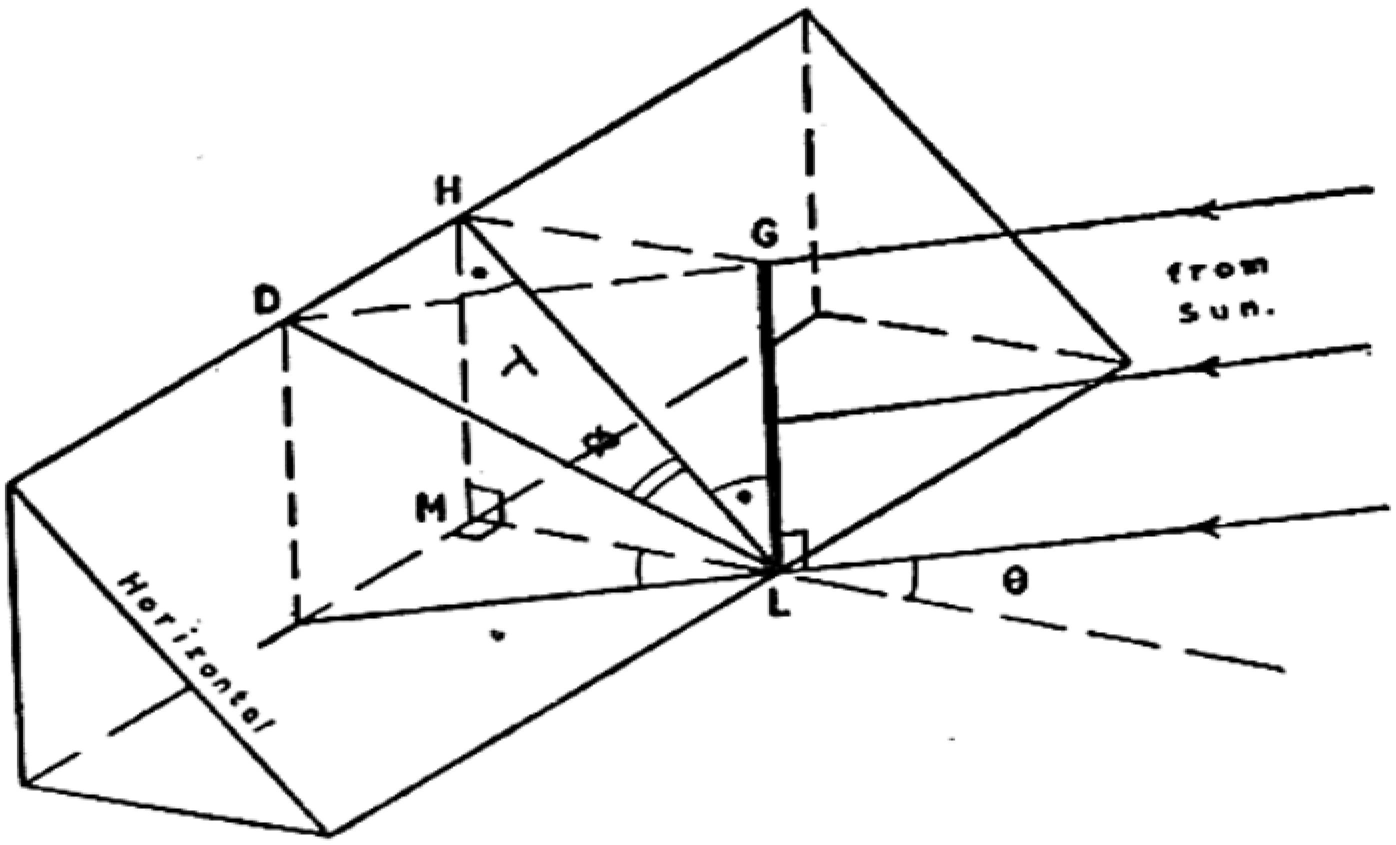

FIG. 192.

3. Show that the construction illustrated in Fig. 193 (next page) is correct.

4. Obtain a formula suitable for a sundial on a vertical wall.

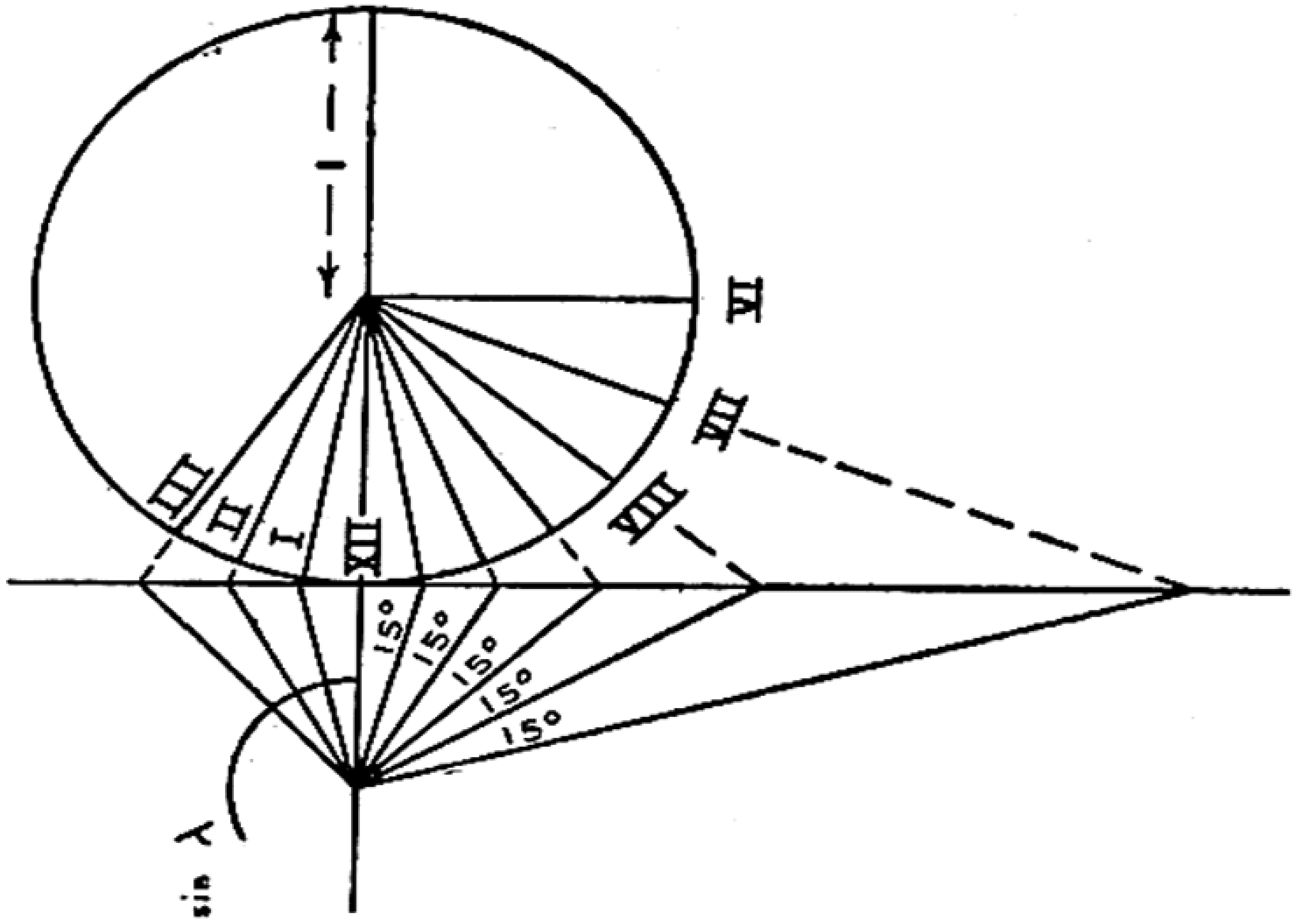

FIG. 193.

N.B. The sundial is only approximately correct as a time-keeper, and may be as much as $\frac{1}{4}$ hour slow in February or $\frac{1}{4}$ hour fast in November, though it is accurate enough for ordinary purposes in the summer.

Time by clock = Time by dial + "Equation of Time." The "equation" is given for each day in Whitaker's Almanack.

In an actual sundial GL is the edge of the "gnomon," or "style."

17. THE CALCULATION OF π

It is proved in more advanced mathematics that, if x is the radian measure of any angle not greater than 45°,

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \ . \ . \ . \text{ and so on.}$$

This is known as Gregory's series, and was discovered in 1670.

If $x = 1$, $\tan^{-1} x = \frac{1}{4}\pi$ and we get

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \ . \ . \ .$$

This series is useless for calculating π because so many terms would have to be taken to get even a rough approximation. Various ingenious dodges have been used since Gregory's time to shorten the work.

1. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{1}{4}\pi$.

[Let $\tan^{-1}\frac{1}{2} = \alpha$ and $\tan^{-1}\frac{1}{3} = \beta$.

Then $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = 1$.]

2. Work out Gregory's series for $\tan^{-1}\frac{1}{2}$ until you come to a term whose value is less than ·0001.

[$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3}(\frac{1}{2})^3 + \frac{1}{5}(\frac{1}{2})^5 - \frac{1}{7}(\frac{1}{2})^7 \ldots$ First make a table of the decimal values of $\frac{1}{2}$, $(\frac{1}{2})^3$, $(\frac{1}{2})^5$, etc., noting that each of these is $\frac{1}{4}$ of the previous term. Next work out the positive terms and add them up, then the negative terms and subtract the two totals. Work systematically, as shown :—

$\frac{1}{2}$	·50000	·50000	
$(\frac{1}{2})^3$	·12500		$\frac{1}{3}(\frac{1}{2})^3$ ·04167
$(\frac{1}{2})^5$	·03125	$\frac{1}{5}(\frac{1}{2})^5$ ·00625	
$(\frac{1}{2})^7$	·00781		$\frac{1}{7}(\frac{1}{2})^7$ ·00112
. . . . .		. . . . . .	. . . . . .]

3. Work out the value of $\tan^{-1}\frac{1}{3}$ similarly, and use the result $\frac{1}{4}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ to calculate π to three places.

4. Prove that $2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4}$ and that

$$\tan^{-1}\tfrac{3}{4} + \tan^{-1}\tfrac{1}{7} = \tfrac{1}{4}\pi.$$

Calculate π from this, using the value of $\tan^{-1}\frac{1}{3}$ already obtained (or preferably finding a few more places).

5. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{1}{4}\pi$.

6. Prove that $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{1}{4}\pi$. [First show that $2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{5}{12}$ and then $2\tan^{-1}\frac{5}{12} = \tan^{-1}\frac{120}{119}$]

This method was used by William Shanks in 1873–4 to calculate π to 707 decimal places.

The last 177 places were in 1946 shown to be incorrect by D. F. Ferguson, who found the value to 800 places.

The following rhyme gives 30 places, the digits being the number of letters in each word :—

Que j' aime à faire apprendre un nombre utile aux sages.
Immortel Archimède, artiste, ingènieur !
Qui de ton jugement peut priser la valeur ?
Pour moi, ton problème eût de pareils avantages.

18. HARMONIC CURVES

1. When a gramophone is sounding a single pure note the groove has the shape of a sine curve, the graph of $y = \sin x$ with the y-scale much smaller than the x-scale. When a note and its octave are sounding together, and are of equal strength, the curve is that of $y = \sin x + \sin 2x$. Draw this by first drawing on the same axes the graphs of $\sin x$ and $\sin 2x$, and adding the ordinates as shown in

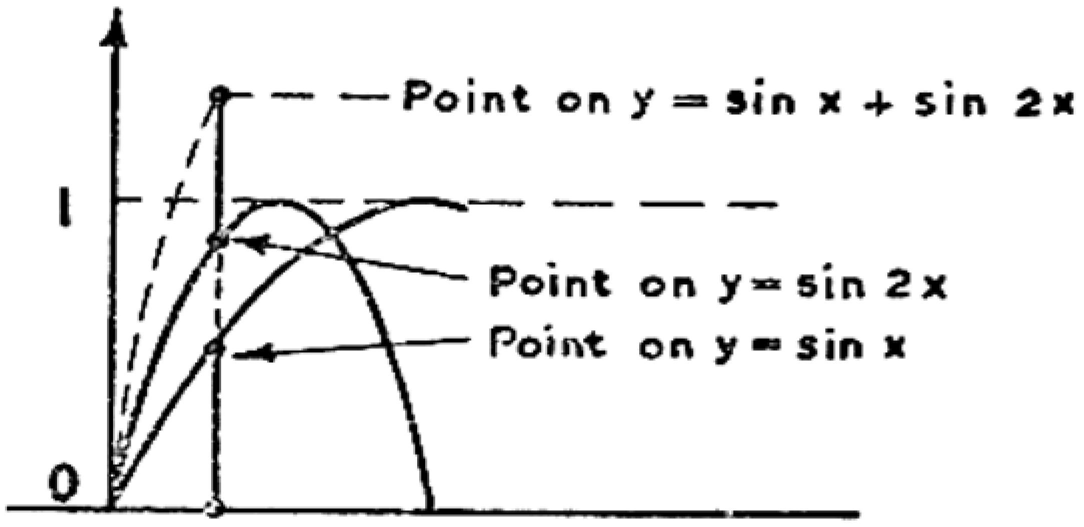

FIG. 194.

Fig. 194. Use compasses or dividers. Scale: for x, 1 inch for 100°; for y, 1 inch for 1 unit.

2. A clarinet sounding a principal note of frequency 200 sounds at the same time a strong 'harmonic' of frequency 600. If this harmonic is half the strength of the principal note the graph of the resultant effect is $y = \sin x + \frac{1}{2} \sin 3x$. Construct this graph by the addition method.

3. A twin-engined aircraft often makes a 'warbling' sound because the speeds of the two engines are not the same. If the speeds of the engines are in the ratio of 4 to 5 the graph of the sound heard is $y = \sin 4x + \sin 5x$. Construct this graph: you will see that the "waves" rise and fall in height.

In music the warbles are known as 'beats'; the number of beats heard per second is the difference between the frequencies of the two notes. A similar phenomenon in radio waves is called 'heterodyning.'

19. THREE PROBLEMS

1. A coffee tin has a paper label, and a ‘ circle ’ is drawn on it, using compasses in the usual way. On unwrapping and flattening (developing) the label the ‘ circle ’ is seen to be an oval. Calculate the dimensions of this oval if the compasses were opened to 3 inches and the tin had a radius of 2 inches.

2. Wrap a sheet of paper round a wooden cylinder. Saw through paper and wood at an angle. Unwrap the paper and its edge will be seen to resemble a sine curve. Prove that it really is a sine curve.

3. Prove that the side elevation of the handrail of a ‘ spiral ’ staircase is a sine curve.

20. COORDINATE GEOMETRY

1. The *Addition Formulæ,* using coordinates. In the figure the circle has unit radius. If $\angle XOP = \theta$ and $\angle XOQ = \phi$, then P is $(\cos\theta, \sin\theta)$ and Q is $(\cos\phi, \sin\phi)$ and this is true for all values of θ and ϕ.

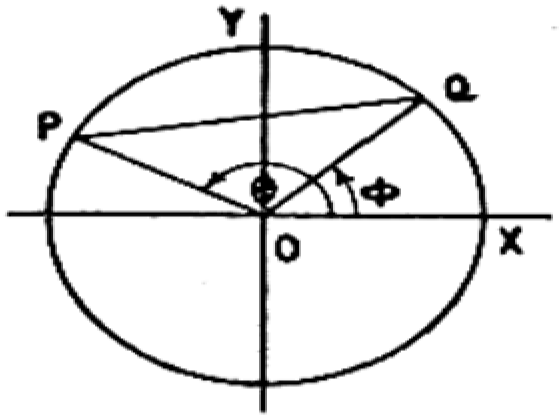

FIG. 195.

(i) Use the cosine formula in $\triangle POQ$ to prove
$$PQ^2 = 2 - 2\cos(\theta - \phi).$$

(ii) Use the coordinates of P and Q to show that
$$PQ^2 = (\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2.$$

(iii) Show that this last expression reduces to
$$PQ^2 = 2 - 2(\cos\theta\cos\phi + \sin\theta\sin\phi).$$

(iv) Comparing the results (i) and (iii) obtain the formula for $\cos(\theta - \phi)$.

(v) Write $-\phi$ for ϕ and obtain the formula for $\cos(\theta + \phi)$.

(vi) Write $(90^\circ + \theta)$ for θ and deduce the formulæ for $\sin(\theta \pm \phi)$.

N.B. In the figure $\theta - \phi < 180^\circ$. If $\theta - \phi$ is between 180° and 360° then the angle in the triangle is $360^\circ - (\theta - \phi)$. But $\cos[360^\circ - (\theta - \phi)]$ is equal to $\cos(\theta - \phi)$, so result (i) is unaffected.

2. The *Linear Equation*, using Coordinates.

(i) Show that the point ($\cos\theta$, $\sin\theta$) is always on the circle whose equation is $x^2 + y^2 = 1$.

(ii) Show that the equation $a\cos\theta + b\sin\theta = c$ is equivalent to the equations $ax + by = c$ and $x^2 + y^2 = 1$, and that the solutions of $a\cos\theta + b\sin\theta = c$ can be found by drawing a straight line and a circle, the radii to the points of intersection giving the required angle.

(iii) Use this method to solve, by drawing, the equations
(α) $6\cos\theta + 8\sin\theta = 5$. ($\beta$) $2\sin\theta + 3\cos\theta = 4$.
(γ) $4\sin\theta - 3\cos\theta = 2$. ($\delta$) $7\cos\theta - 4\sin\theta = 8$.
Compare with the solutions for Ex. 2, 3, 4, p. 132.

21. Direction-Cosines

In solid geometry the direction of a line OP is specified by the angles α, β, γ which it makes with the axes, or rather by the cosines of the direction angles, which are called the direction cosines of OP.

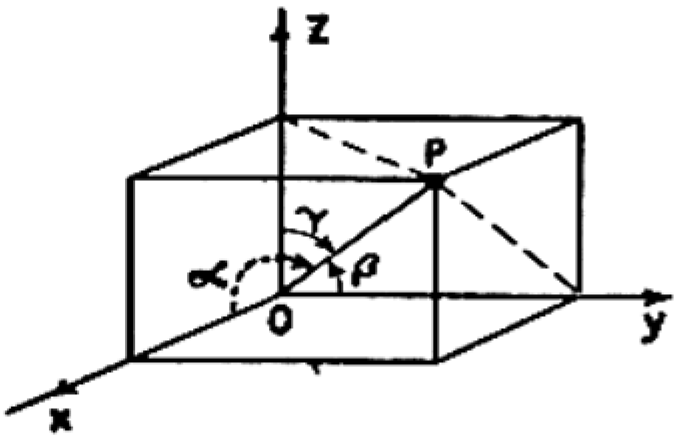

FIG. 196.

1. If P is (x, y, z) and OP is r, show that $x = r\cos\alpha$, etc. Prove that $x^2 + y^2 + z^2 = r^2$ and deduce that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

2. If OP is r and the 'direction cosines' of OP are l, m, n, what are the coordinates of P?

3. If the direction cosines of OP are proportional to 3, 4, 5, what are their actual values? (They are $3k$, $4k$, $5k$, and the sum of their squares is 1.)

4. If two lines have direction angles α, β, γ; $\alpha_1, \beta_1, \gamma_1$, prove that the cosine of the angle between them is $\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1$.

(If OP, OQ are the lines, apply the cosine formula to $\triangle$POQ and also get PQ from the formula

$$PQ^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

Use the result of Ex. 1.)

5. If $\gamma = 90°$ show that $\alpha + \beta = 90°$. What plane trigonometry result is obtained from the result of Ex. 4?

22. PLANS AND ELEVATIONS

1. Fig. 197 shows the plan and elevation of a line AB. Calculate the angle which it makes with the horizontal plane and also that which it makes with the vertical plane shown in the diagram.

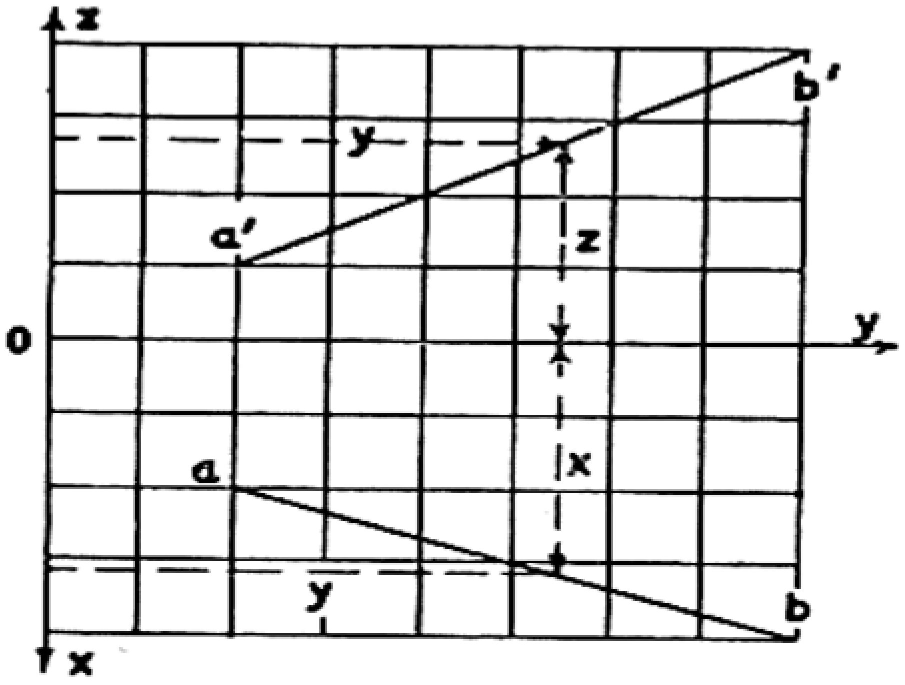

FIG. 197.

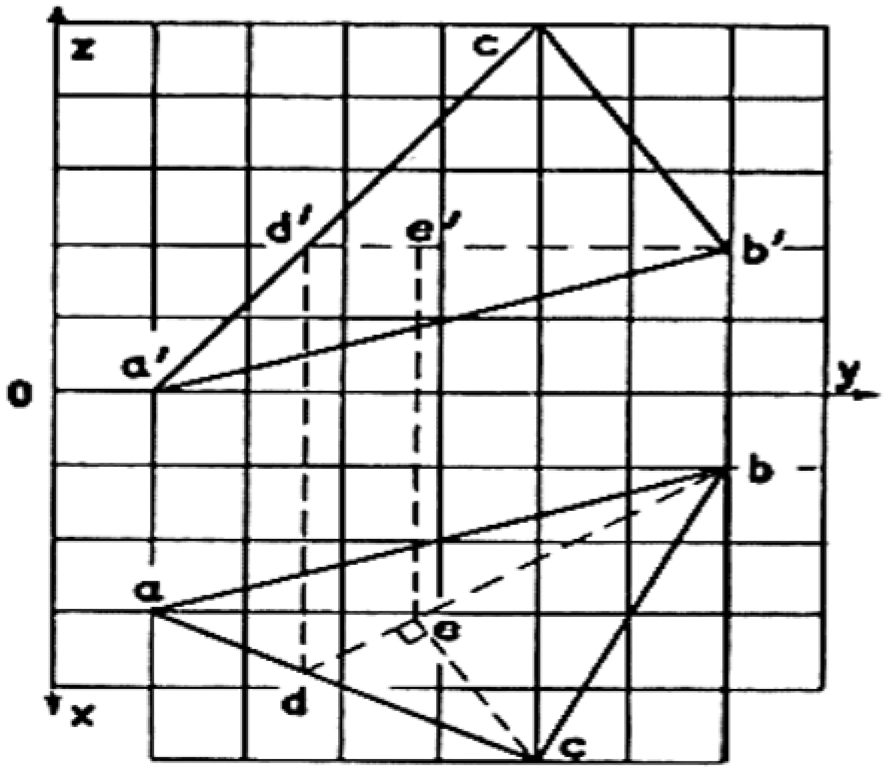

FIG. 198.

Show that the length of AB is 7 units and that if (x, y, z) lies on it, then

$$\frac{x-2}{2} = \frac{y-2}{6} = \frac{z-1}{3}.$$

Show that AB makes with the vertical the angle $\sec^{-1} 7/3$ and find this angle in the tables.

2. Fig. 198 shows the plan and elevation of a triangle ABC. Calculate the lengths of its sides and the angle which each of them makes with the horizontal.

Explain why *bd* is the plan of a horizontal line and *ec* ($\perp$ *bd*) that of a line of greatest slope and by measurement show that the angle which the plane of the triangle makes with the horizontal is in the neighbourhood of 64°. Prove that the median drawn from A makes with the horizontal an angle whose tangent is 0·7, and find the corresponding result for the median drawn from C and the angle itself.

3. A right block rests on the ground with its base corners at the points (2, 3), (3, 5), (7, 3), (6, 1), and its height is 2 units. Draw its plan and elevation. Calculate the angles which its diagonals make with the horizontal plane.

23. Simple Harmonic Motion

If OP (in Fig. 199) rotates with constant speed the point N moves up and down the vertical diameter with what is called *simple harmonic motion* (S.H.M. for short). While OP makes one revolution N makes one *oscillation*, *i.e.*, moves from O to B, down through O again to D and back to O.

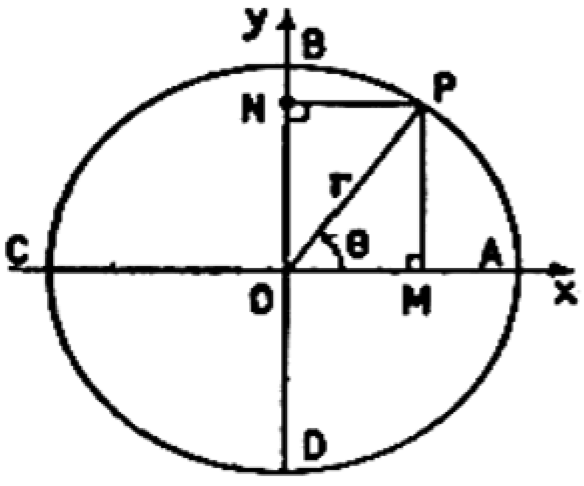

FIG. 199.

The point O is called the *mean position* of N.

The distance OB or OD is the *amplitude* of the motion.

The time of an oscillation is the *period* of the motion.

If ON $= y$ and OP $= r$, then $y = r \sin \theta$, and r is the amplitude.

If the angular velocity of OP is ω radians per second, then the period is $2\pi/\omega$, since the period is the time of one revolution of OP.

If the time taken to move from OA to OP is t secs., then $\theta = \omega t$, and the position of N after t secs. is given by $y = r \sin \omega t$. The motion of M is exactly similar to that of N.

The motion of most vibrating bodies is either exactly or approximately simple harmonic, and so can be described by formulæ containing the sine of an angle proportional to the time. The shape of a sine curve suggests a connection with waves, and in fact the particles of air in a sound-wave or of water in surface ripples move with S.H.M.

1. In Fig. 199, what is the speed of P ? At what point is the speed of N equal to that of P ? What is the maximum speed of N ?

2. If the period of N is 5 sec. and its amplitude 2 ft., what is its maximum speed ? [First find ω.]

3. The middle point of a 'cello string making 300 vibrations per second moves in S.H.M. with an amplitude of $\frac{1}{4}$ inch. Find its maximum speed in ft. per sec.

4. In a certain harbour the interval between consecutive high tides is 12 hours, and the depths at high and low water are 30 ft. and 20 ft. Find a formula for the depth of water h hours after the depth was 25 ft. and increasing, assuming the motion to be S.H. [Here the amplitude is

5 ft., and the period 12 hours, so that $\omega = 2\pi/12$ radians per hour.]

5. If the depth of water in a dock h hours after mid-day is given by $d = 20 + 4 \sin 24h^\circ$ find (i) the greatest and least depths, (ii) the time of the next high water, (iii) the time when the water is next 18 ft. deep.

6. Imagine a tunnel bored along the earth's axis. A body allowed to fall down this would (ignoring air resistance) oscillate from one end to the other, a distance of 8000 miles, with S.H.M., the period being about $1\frac{1}{2}$ hours. If the body was at the centre of the earth at mid-day and moving northwards, where would it be at 1 p.m., and what is its greatest speed in miles per hour? [Find ω, and show that $\mathrm{ON} = 4000 \sin 4\pi h/3$.]

7. A pendulum 100 ft. long has a period of 11 secs. If the 'bob' has an amplitude of 5 ft. and the motion is assumed to be S.H.M. in a straight line, find the maximum speed.

8. Draw a tangent at P, meeting Ox at Q, to represent the velocity of P. Resolve this parallel and perpendicularly to Oy and show that the velocity of N is $\omega r \cos \omega t$.

24. Alternating Current

The 'grid' electricity supply is described as "230 volt, 50 cycle." A graph of current or voltage against time is a sine curve having 50 waves or cycles per second.

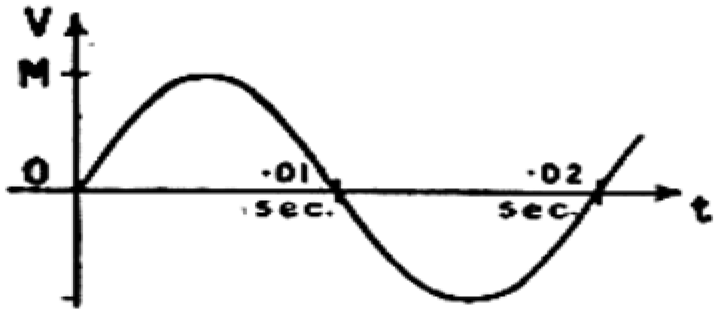

FIG. 200.

1. If M is the maximum voltage, show that $\mathrm{V} = \mathrm{M} \sin 100\pi t$ fits the curve.

2. What is the average voltage? Why is alternating current useless for electroplating?

3. The heat produced in, say, an electric fire is proportional to the square of the voltage applied. What is the average of the *square* of the voltage?

[$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$. Consider the average value of $\cos 2x$.]

4. The 230-volt supply is so called because the average of the square of the voltage is 230^2, so that it produces the same heat as 230-volt *direct* current. What must be the value of M ?

5. 230 volts is the root-mean-square (R.M.S.) of the voltage, because it is the 'square root of the average of the square of the voltage.' Show that the maximum voltage is $\sqrt{2}$ times the R.M.S. voltage, and find for what fraction of the time the actual voltage is above the R.M.S. value.

[This is the same as finding when $\sin\theta$ is numerically greater than $1/\sqrt{2}$.]

6. What is the maximum voltage if the advertised voltage is 1100 ?

[It should now be clear why a worse shock is obtained from A.C. than from D.C. of the same advertised voltage.]

25. Polar Coordinates

The position of a point P can be given by its distance r from a fixed point O and the angle θ which OP makes with a fixed line OX.

The values of r and θ are called the *polar coordinates* of P, and O is called the *pole*.

Many curves are described more conveniently by the use of polar coordinates than by means of the ordinary (x, y) or *cartesian* coordinates.

Take, for example, the polar equation $r = \cos 2\theta$. Draw through O a set of lines at intervals of, say, 15°. Mark a point on each 'radius,' choosing a suitable scale.

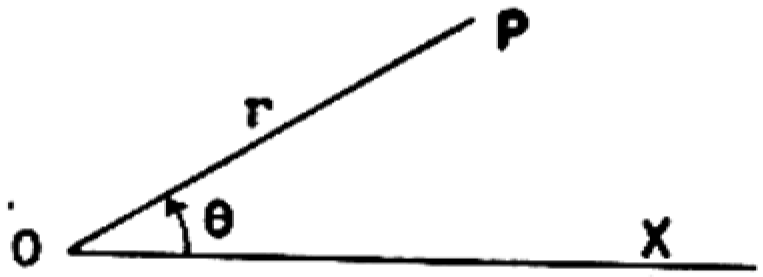

FIG. 201.

θ	0	15°	30°	45°	60°	75°	90°	105° . . .
$\cos 2\theta$	1	·87	·5	0	— ·5	— ·87	— 1	— ·87 . . .

A negative value for r means that the distance must be measured in the opposite direction, *i.e.*, along the radius produced through O.

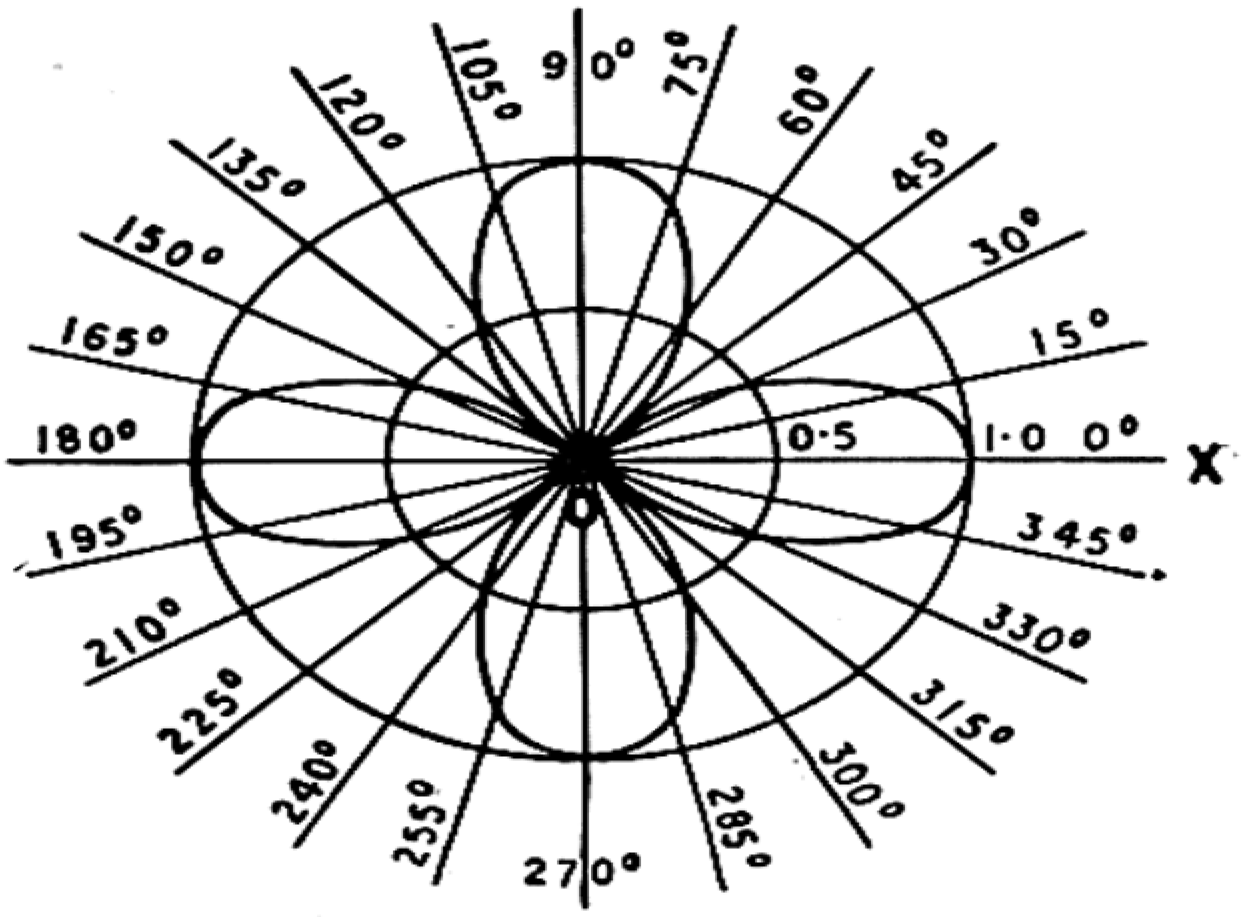

FIG. 202.

Draw a fair curve through the points : it is called a Rose-curve.

Plot the following :—

1. $r = \cos\theta$
2. $r = \sec\theta$
3. $r = 1 + \cos\theta$ (a cardioid) } These can be 'built-on' to No. 1.
4. $r = \frac{1}{2} + \cos\theta$ (a limaçon) }
5. $r = \sin 3\theta$ (a rose-curve)
6. $r = \frac{1}{60}\theta$ (an Archimedean spiral)
7. $r^2 = \cos 2\theta$, *i.e.*, $r = \pm\sqrt{\cos 2\theta}$ (a lemniscate, or 'ribbon')
8. $r = 10^{\theta/1000}$ or $r = \text{antilog}\,\frac{1}{1000}\,\theta$. (An equiangular spiral ; snail shell)

26. SOME CURIOUS GRAPHS

1. $y = x + \sin x$ [Use radians ; take 1 in. for π radians, 1 in. for y.]
2. $y = x \sin x$
3. $y = \dfrac{\sin x}{x}$ [Take $y = 1$ when $x = 0$.]
4. $y = \sin\dfrac{3600^\circ}{x}$ [1 in. for 10 units.]
5. $y = \dfrac{\sin x}{2^x}$

6. How many points can you find on the following 'curves'?

(i) $\dfrac{\sin^{-1} x}{\sqrt{x-1}}$ (ii) $\sqrt{x-1} \,.\, \sin^{-1} x$ (iii) $\sqrt{x^2-1} \,.\, \sin^{-1} x$

27. Two Unexpected Equilateral Triangles

This is a comparatively recent geometrical theorem. If equilateral triangles are described outwardly on the sides of *any* triangle, their centroids form an equilateral triangle. Similarly if the equilateral triangles are drawn so as to overlie the original triangle their centroids form an equilateral triangle.

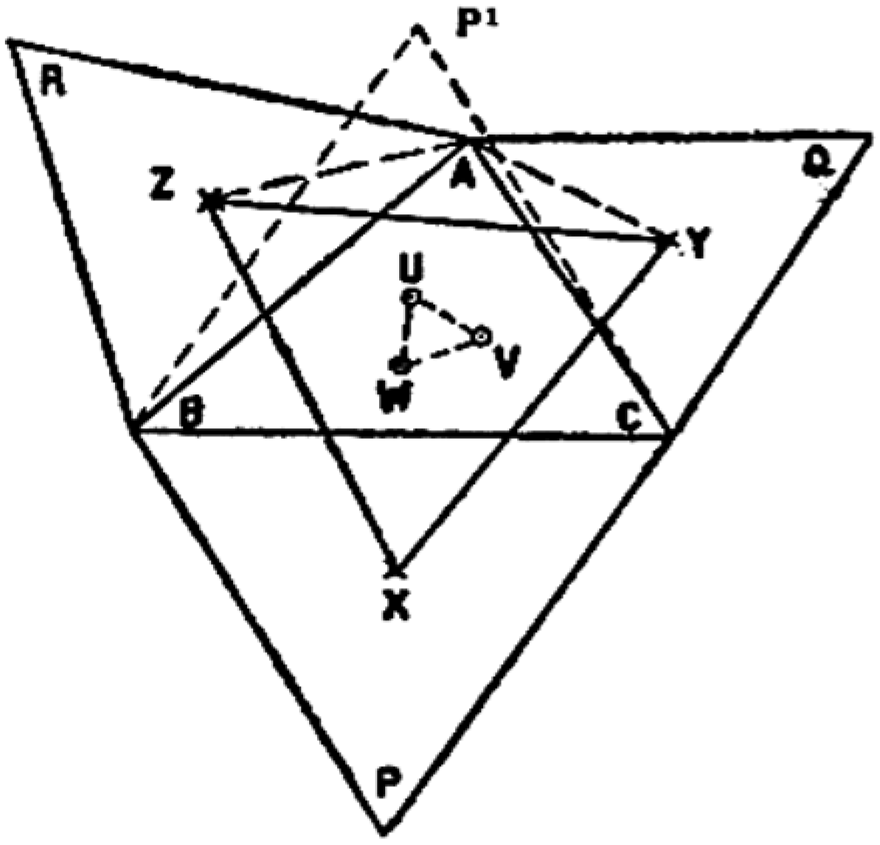

FIG. 203.

The centroid of a triangle is at the intersection of the medians. In the case of an equilateral triangle the medians also bisect the angles.

In the figure, X, Y, Z are the three centroids. Join AZ, AY, and XY, YZ, ZX.

1. If b is the side of an equilateral triangle ACQ, what are (i) AY, (ii) the area ACQ. Leave the results in terms of $\sqrt{3}$.

2. What is $\angle$ZAY?

3. Apply the cosine formula to $\triangle$AZY to find ZY^2.

4. Show that (i) $2\cos(A + 60°) = \cos A - \sqrt{3}\sin A$, (ii) $bc \cos A = \frac{1}{2}(b^2 + c^2 - a^2)$.

5. If Δ is the area of $\triangle$ABC, deduce that

$$ZY^2 = \tfrac{1}{6}(a^2 + b^2 + c^2 + 4\sqrt{3} \,.\, \Delta).$$

The symmetry of this expression shows that we should get the same result for YX^2 and XZ^2, so that $\triangle$XYZ is equilateral.

6. If the centroids of the "inward" set of equilateral triangles are U, V, W, prove $VW^2 = \frac{1}{6}(a^2+b^2+c^2-4\sqrt{3} \,.\, \Delta)$.

7. Using the result of 1. (ii), find the areas E, F of the two equilateral triangles and prove $(E - F) = \Delta$.

8. Since $\angle ZAB = 30° = \angle WAB$, and $AZ = AW$, then $\triangle AZN$ is congruent to $\triangle AWN$. Similarly $\triangle AYM = \triangle AVM$. Prove that the vector sum of ZY and WV is BC, or $\overline{ZY} + \overline{WV} = \overline{BC}$.

$$[\overline{ZY} = \overline{ZA} + \overline{AY} = \overline{ZN} + \overline{NA} + \overline{AM} + \overline{MY},$$
$$\overline{WV} = \overline{WA} + \overline{AV} = \overline{WN} + \overline{NA} + \overline{AM} + \overline{MV}.$$

Add, and remember $BA = 2NA$, $AC = 2AM$.]

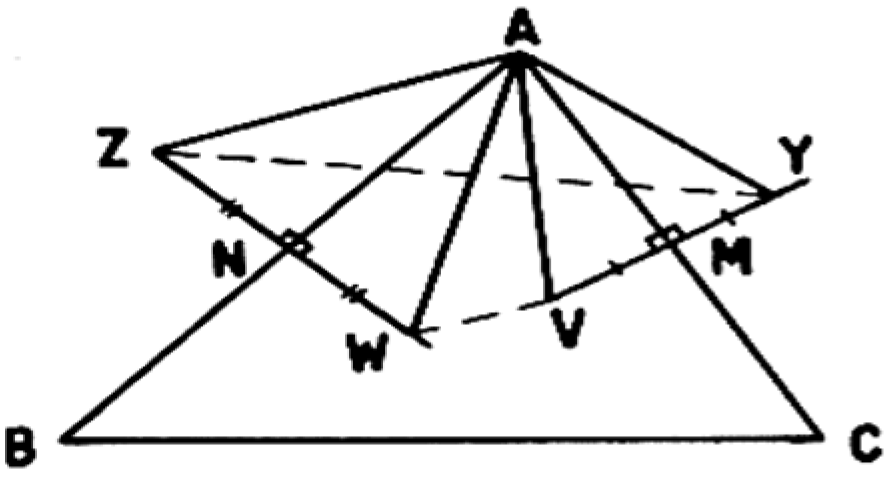

Fig. 204.

The result of No. 8 and the two similar results for CA and AB are used in three-phase work in electrical engineering. This shows how impossible it is to say whether a mathematical theorem, however fantastic, is ‘ useful ’ or not.

28. The Cycloid

A bicycle wheel moves along without slipping : what is the path of a drawing pin stuck in the tread ?

The circle centre C represents the wheel and P the pin. At first the pin was on the ground at O and C at A. In the

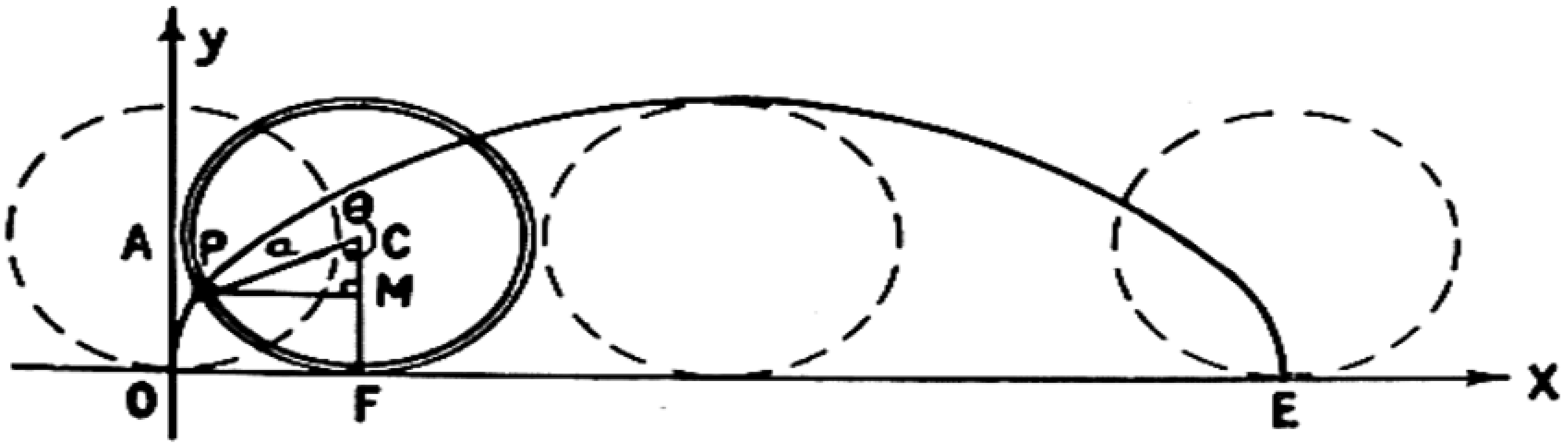

Fig. 205.

position shown the radius CP has turned through an angle θ from its first (vertical) position.

1. If the radius of the wheel is a, and $PM \perp CF$, express in terms of a and θ (i) OF (= arc PF), (ii) PM, (iii) CM. [Take θ to be in radian measure.]

2. What are the coordinates of C ?

3. If P is (x, y), show that $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.

4. If the pin touches the ground again at E, what is the distance OE ?

5. Sketch the curve, either from the results of No. 3 or by drawing several positions of the wheel.

[This curve is sometimes used for the arch of a bridge.]

29. THE 'UNWINDING' CURVE

The cycloid is formed when a circle rolls on a fixed line. What curve is formed when a line rolls on a circle ?

An example of this curve is the path of the end of a cotton thread being unwound from a reel, the thread being kept taut. In Fig. 206, the end P of the thread PF was originally at A.

1. When $\angle AOF = \theta$ (radians), what is the length FP ?
2. Express in terms of θ (i) PM, (ii) FM.
3. If P is (x, y) show that $x = a(\cos\theta + \theta\sin\theta)$
   $y = a(\sin\theta - \theta\cos\theta)$.

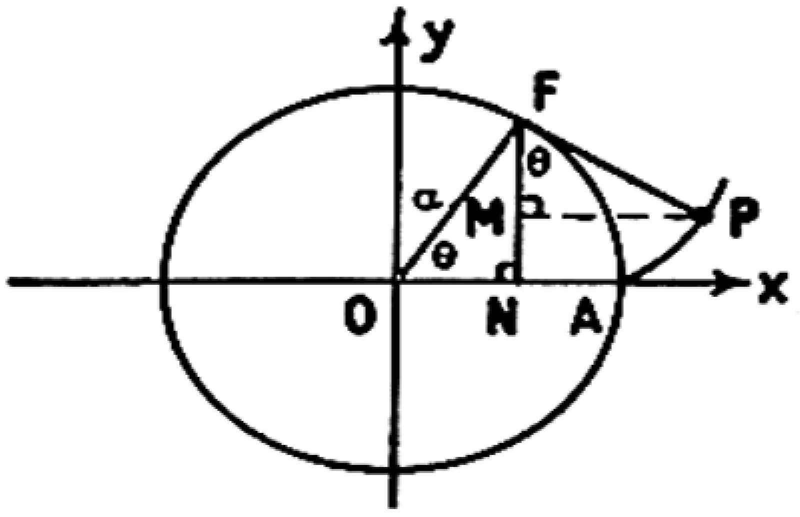

FIG. 206.

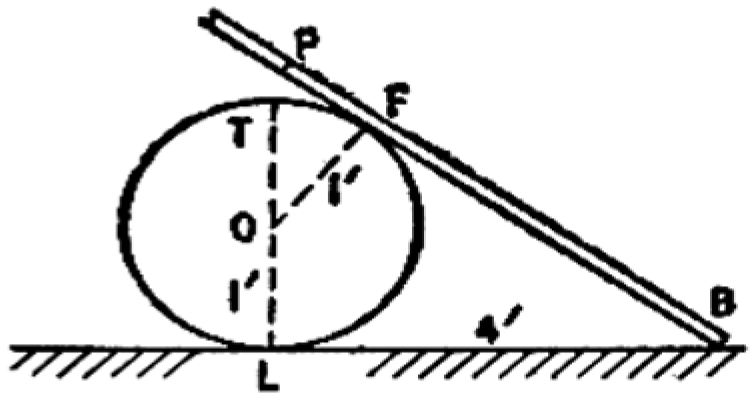

FIG. 207.

4. Show that $x^2 + y^2 = a^2(1 + \theta^2)$.
5. What is the value of θ when P is next on AO (produced) ? Use the result of Ex. 3.
6. A see-saw is made from a plank AB and a cylindrical log 2 ft. in diameter. The ends A and B touch the ground at points 8 ft. apart. What is the length of the plank ? [Only part of the plank is shown. P is its mid-point. Find $\angle FBL$ and arc TF. Fig. 207]

[The technical name of the path of the end P of the thread, or of the mid-point P of the plank, is the involute of a circle, which means the 'unwound' curve. It is used in the design of gear-teeth.]

30. EPICYCLOIDS

If a circle rolls round the outside of another circle a point on the circumference describes an *epicycloid*. In the figure, the point P starts from A, and the centre of the rolling circle at B. If the circles are equal, P traces out a heart-shaped curve called a *cardioid*, or one-cusped epicycloid.

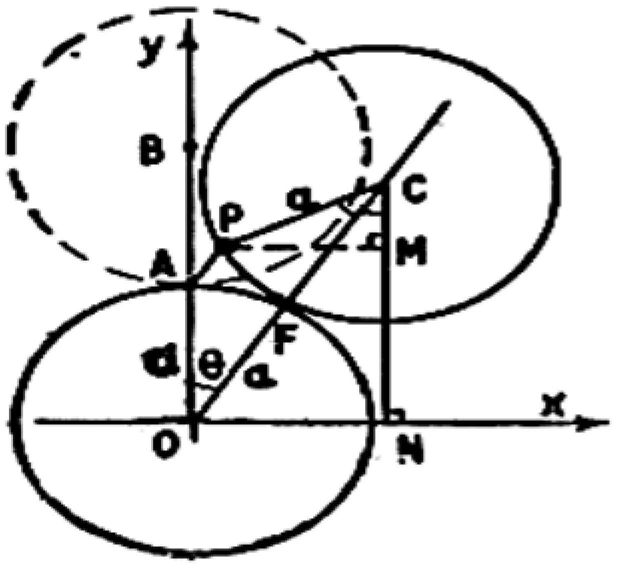

FIG. 208.

1. Since arc AF = arc PF, ∠PCO = θ. Show that ∠PCM = 2θ and express (i) PM, (ii) CM in terms of a and θ.

2. What are the coordinates of C ?

3. If P is (x, y), show that $x = a(2 \sin \theta - \sin 2\theta)$, $y = a(2 \cos \theta - \cos 2\theta)$.

4. Show that $x^2 + y^2 = a^2(5 - 4 \cos \theta)$.

5. If the rolling circle has radius a and the fixed circle radius $3a$, show that the curve obtained has 3 cusps or ‘ beaks,’ and find equations for x and y as in No. 3.

[The teeth of gear-wheels are cut to the shape of a suitable epicycloid, so that the teeth of two wheels engage with a rolling and not a scraping motion.]

31. HYPOCYCLOIDS

When a circle rolls *inside* another circle a point on its circumference describes a *hypocycloid*. Three-speed gears for bicycles have wheels arranged to move in this way. In the figure the radius of the small circle is a and of the big circle $3a$.

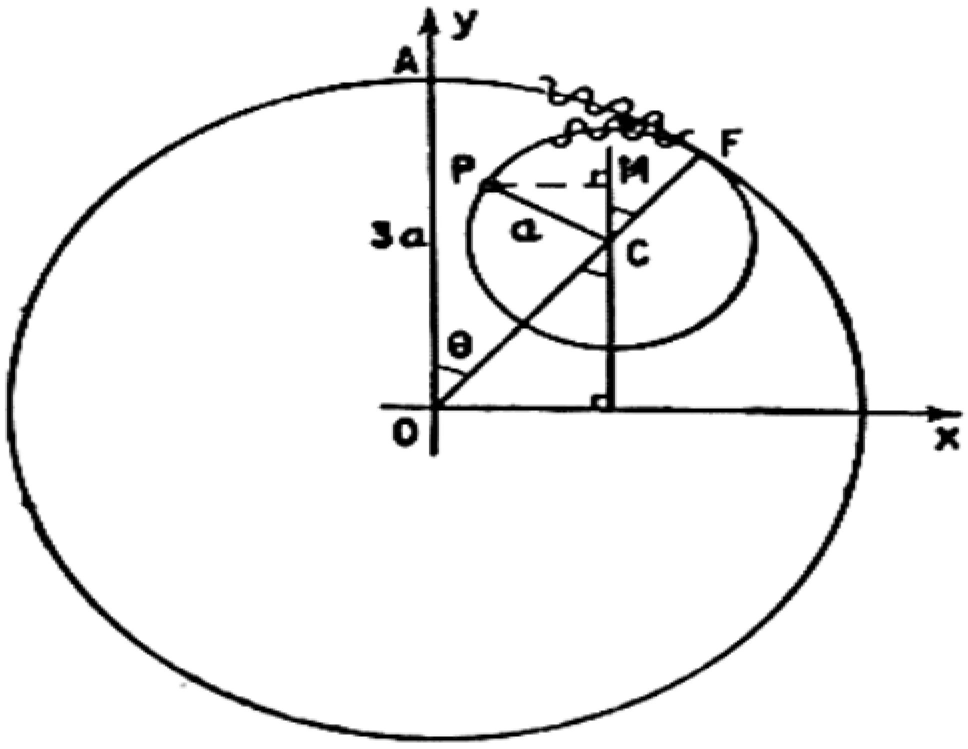

FIG. 209.

1. By considering the equal arcs AF, PF show that ∠PCF = 3θ and ∠PCM = 2θ.

2. What are the coordinates of (i) C, (ii) P ?

3. How many ‘ cusps ’ will the curve have ? Sketch the curve.

4. If the radii are $4a$ and a, show that

$$x = a(3\sin\theta - \sin 3\theta),\ y = a(3\cos\theta + \cos 3\theta)$$

and, using the formulæ for $\sin 3\theta$ and $\cos 3\theta$, that $x = 4a\sin^3\theta$, $y = 4a\cos^3\theta$. Hence prove $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (4a)^{\frac{2}{3}}$. This curve is a four-pointed 'star,' and is called the '*astroid*,' or four-cusped hypocycloid.

5. If the radii of the circles are $2a$ and a, show that $x = 0$, so that P simply moves down and up the diameter AOB. The two cusped hypocycloid is a mere straight line.

32. DIFFERENTIATION OF cos θ AND sin θ

[For those who know that, if s is distance, $\dfrac{ds}{dt}$ is velocity and that $\dfrac{ds}{dt} = \dfrac{ds}{d\theta}\cdot\dfrac{d\theta}{dt}$].

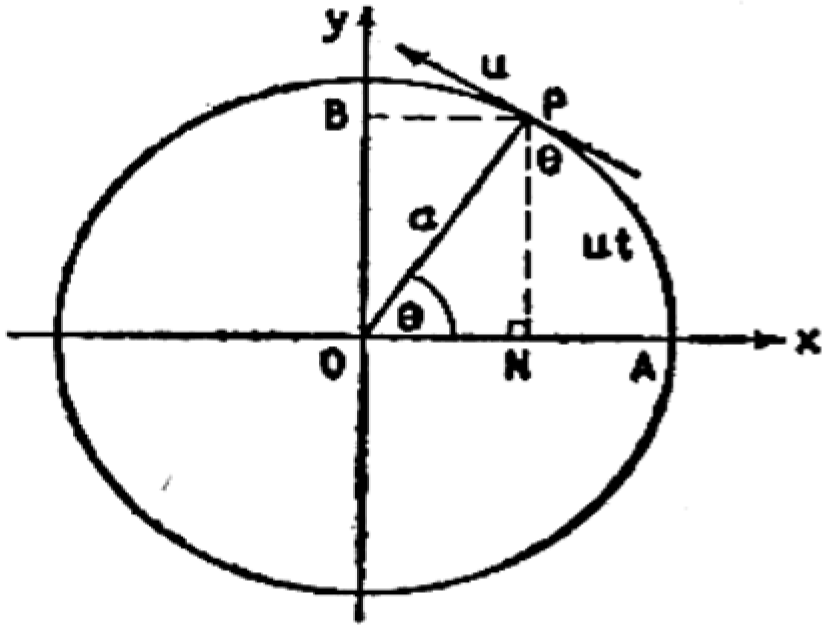

FIG. 210.

P moves round a circle of radius a with velocity u starting at A (Fig. 210) when $t = 0$. Then arc AP $= ut$.

But *if* θ *is in radians* arc AP $= a\theta$

$$\therefore\ a\theta = ut \qquad \therefore\ \frac{d\theta}{dt} = \frac{u}{a}.$$

If ON $= x$, PN $= y$, then $x = a\cos\theta$ and $y = a\sin\theta$.

Also the velocities of P parallel to the axes of x and y are

$\dfrac{dy}{dt} = u\cos\theta$ and $\dfrac{dx}{dt} = -\,u\sin\theta$ ($-$ since x is decreasing).

$$\therefore\ \frac{dy}{dt} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dt} \quad\text{and}\quad \frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt}\ \text{give}$$

$$u\cos\theta = \frac{dy}{d\theta}\frac{u}{a} \quad\text{and}\quad -\,u\sin\theta = \frac{dx}{d\theta}\cdot\frac{u}{a}.$$

$$\therefore\ \text{If } y = a\sin\theta,\ \frac{dy}{d\theta} = a\cos\theta$$

and if $x = a\cos\theta$, $\dfrac{dx}{d\theta} = -\,a\sin\theta$.

Thus $\dfrac{d}{d\theta}\sin\theta = \cos\theta$ and $\dfrac{d}{d\theta}\cos\theta = -\sin\theta$,
always provided that θ is the radian measure of the angle. (This is not the usual way but it shows very clearly why θ must be in radians.)

1. If $\theta = z$ degrees, show that $a\theta$ is replaced by $\pi az/180$ and deduce $\dfrac{d}{dz}\sin\theta$ and $\dfrac{d}{dz}\cos z$.

2. What difference does it make if A is not the starting point when $t = 0$, but P is at A when $t = t_1$?

3. What is the derivative of $4\cos\theta + 3\sin\theta$? Does this agree with what you get when you make

$$4\cos\theta + 3\sin\theta = 5\cos(\theta - \alpha)?$$

33. SOLUTION OF TRIANGLES BY DIAGRAM

Any one triangle can of course be solved by drawing a diagram, and careful work on a large scale may give three-figure accuracy.

There are, however, several ways in which *one* diagram may be used to solve *all* triangles or at least several large classes of triangles. Three of these will be briefly discussed. The ideas underlying these are instructive, and they should be interesting to those who like drawing. Their defect is that very careful drawing on a large scale is needed to ensure three-figure accuracy.

I. A diagram (of the type called a 'nomogram') for the sine-formula.

In Fig. 211, AB + EF = 2CD, since BD = DF. Hence, if the scales are logarithmic (*i.e.*, a point ·301 units from B is marked 2) and if the scale on the middle line is half that on the others, we get for the *marked* lengths

$$CD = AB \times EF,$$

i.e., reading at C = (reading at A) × (reading at E).

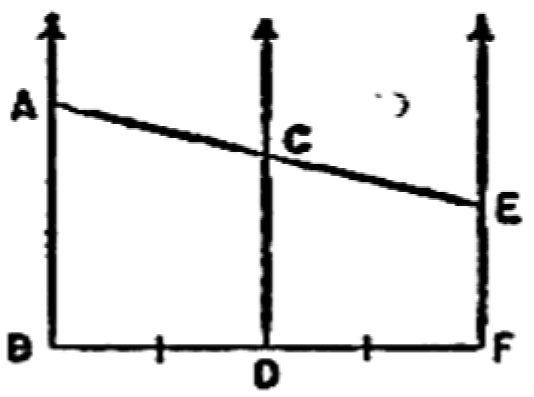

FIG. 211.

Fig. 211 gives the sine formula if AB is log sin A, CD is log a, and EF is log 2R, since $a = 2R\sin A$.

Note that log sin 90° = 0,

$$\log\sin 30^\circ = \log\cdot5 = \bar{1}\cdot699 = -\cdot301,$$

so that the distance from 90° to 30° is ·301 on a suitable scale. (See Fig. 212).

If A on the left-hand scale is joined to a on the middle one, and B on left to b on middle, these lines should meet on the right-hand line at a point showing 2R.

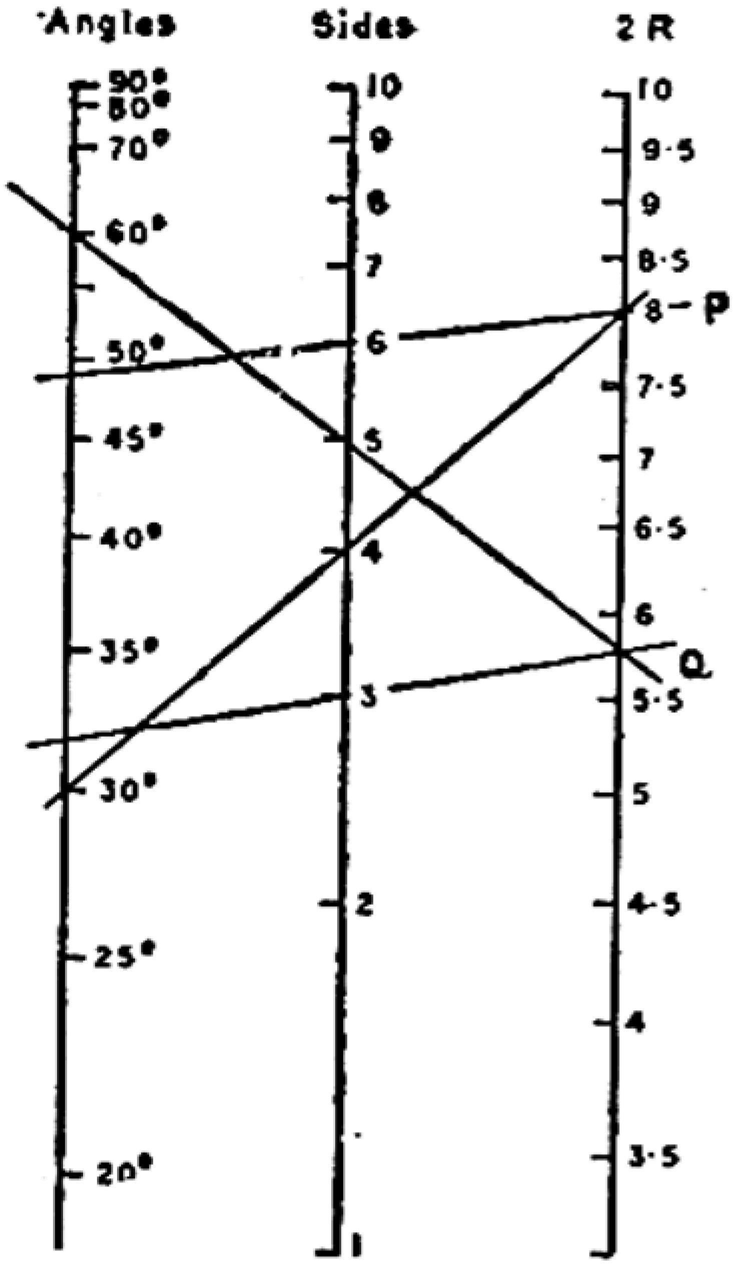

FIG. 212.

The lines meeting at P show 4/sin 30° = 6/sin $48\frac{1}{2}$° and those meeting at Q show 5/sin 60° = 3/sin 31°.

If one of the four a, b, A, B, is unknown this construction finds it.

1. If 4 is opposite 30°, what is 7 opposite? (Join P to 7 on middle scale.)
2. If 5 is opposite 60°, what is 4 opposite?
3. If 6 is opposite 60°, draw lines to show that 40° is opposite 4·4 or 4·5.
4. Explain why, if the scale for the sides is doubled and the scale for the angles turned upside down, then a,A and b,B lines will intersect half way between the a,b and A,B lines.

N.B. The distance of the graduation for θ from the top of the line is log cosec θ on some suitable scale, *e.g.*, 10in. for one unit. This places 30° at 0·301 units from the top. The scale is found on the back of most slide-rules.

II. The diagram, shown partly drawn in Fig. 213, is also based on the sine formula, and is suitable for all cases of solution except S.S.S. (to which it can be adapted clumsily). Squared paper (10 in. square) is marked with decimals and also with a scale of angles from 0° to 90° of which these decimals are the sines (see Fig. 213).

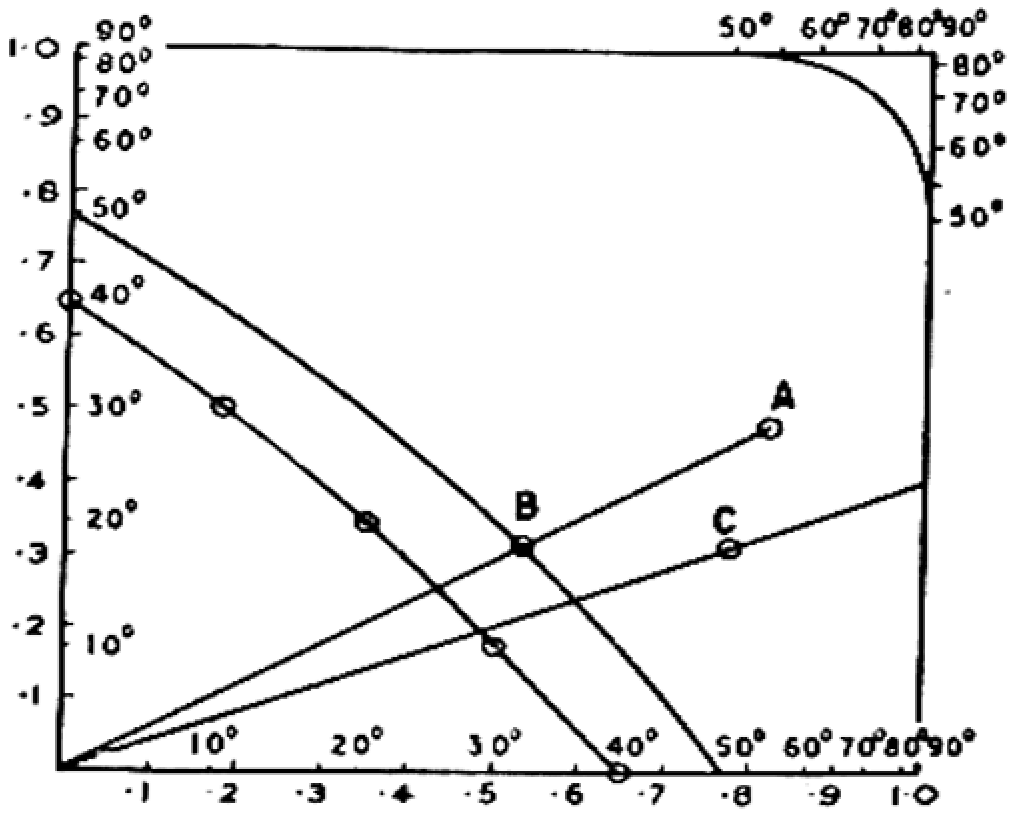

FIG. 213.

(These graduations should be ruled in coloured ink right across the paper.) A set of curves is then drawn of which those for 40° and 50° are shown. The first goes through (0°, 40°), (10°, 30°), (20°, 20°), (30°, 10°), (40°, 0°) and similarly for the others. So these curves are very easily drawn.

The diagram shows how to solve a triangle in which sides 82, 48 contain 130°, as explained below.

1. How does A represent 82,48 ?

2. Why are the two angles shown by B such that their sines are as 82 : 48 ?

(The sum of these angles is $50° = 180° - 130°$ so they are the other angles of the triangle.)

3. C being taken so that its x is sin 130° and its y that of B, explain why the third side would be given by the x of the point whose y is that of A on the line shown through C. (This point being outside the paper, it is easy to take the point on the line where the y is half that of A and to double the corresponding x.)

4. If two angles of a triangle are 32° and 18° (shown roughly at B), and if 82 is opposite 32°, show that A provides the side opposite 18°.

5. If in the diagram the origin is joined to the point where the curve from (90°, 50°) to (50°, 90°) cuts $x=0{\cdot}9$, what triangle would be solved by the point where this line cuts $x=0{\cdot}3$?

III. The Triangle Chart. (For those who know what graphs are given by $x + y = 1$ and $x^2 + y^2 = 1$.).

This involves the idea that the *shape* of a triangle depends on *two* numbers which may be the ratios to the largest side (taken as 1) of the other two sides (taken as x, y where $x > y$). Then to any triangle corresponds a point (x, y).

1. In Fig. 214 show that (the sides being 1, x, y in *descending* order), every triangle has its shape represented by a point of $\triangle$ABD uniquely. (Remember $x + y > 1$.)

2. Show that points on AB represent isosceles triangles with their larger sides equal and points on BD isosceles triangles with their smaller sides equal.

3. Show that points on AD represent nugatory triangles of which two sides are equal to the third.

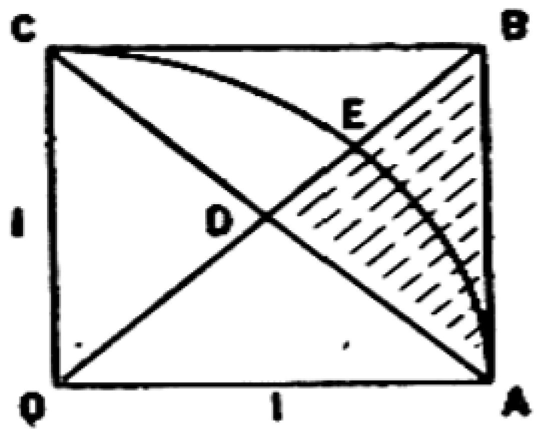

Fig. 214.

4. Show that points on the circular arc EA (centre O) represent right-angled triangles; those in EAD obtuse-angled triangles and those in EAB acute-angled triangles.

5. If CB is produced to F so that BF = CB, show that if the sides of a triangle are taken as x, 1, y in descending order, there is a unique correspondence between the shape of any triangle and a point of $\triangle$ABF.

ANGLES AND AREA SHOWN ON THE DIAGRAM

Just as points on $x^2 + y^2 = 1$ represent right-angled triangles, so points on $y^2 - x^2 + 2x \cos \gamma = 1$, got from the cosine formula, represent triangles of which the least angle is γ and points on $x^2 - y^2 + 2y \cos \chi = 1$ triangles whose middle angle is χ. These curves are rectangular hyperbolas and are the plain curves in Fig. 215. The second set are those which converge at (1, 0).

It is also possible to draw a set of curves (dotted in the figure) to show triangles whose areas are given as decimals of the square on the largest side.

By using the data to mark the point corresponding to the required triangle all cases of solution can be tackled (except the S.A.S. case when the unknown side is the largest, for which the diagram can be adapted, clumsily).

It needs a 20-in. unit and angle-curves drawn for each degree to secure three-figure accuracy for most triangles.

The diagram also affords information as to how one element of a triangle can vary if other elements are given, as in Ex. 4 to 8.

1. *Get rough answers from* ***Fig.*** **215.**

Show that if two angles of a triangle are each 50° or if one is 65° and another 40° the area is approximately ·3 of square on largest side.

2. Use the diagram to solve roughly the triangles

(i) Largest side 10 ; two angles 30° and 55°.

(ii) Two largest sides 40, 32 containing 40°. [Here $x = \cdot 8$]

(iii) Angles 35°, 65°, 12′ opposite the 65°.

[The angles make $x \simeq \cdot 92$. This corresponds to 12.]

3. How does the diagram show that the 35°, 60°, 85° triangle is a quarter of the square on the largest side ?

4. The least angle of a triangle is 35° ; between what limits will the ratio of its least to its greatest side lie ?

5. The least side of a triangle is ·7 of its largest side. Show that its least angle is between 40° and 45°, and find the range of its greatest angle.

6. If the area of a triangle is two-tenths of the square on its

greatest side, show that its least angle is not greater than 38°.

7. The largest side of a triangle is 10″. Its least angle is 55°. Show that its area lies between 36 and 41·5 square inches.

8. If the least angle of a triangle is 45°, the altitude to the largest side is at least half of it and cannot be more than ·72 of it.

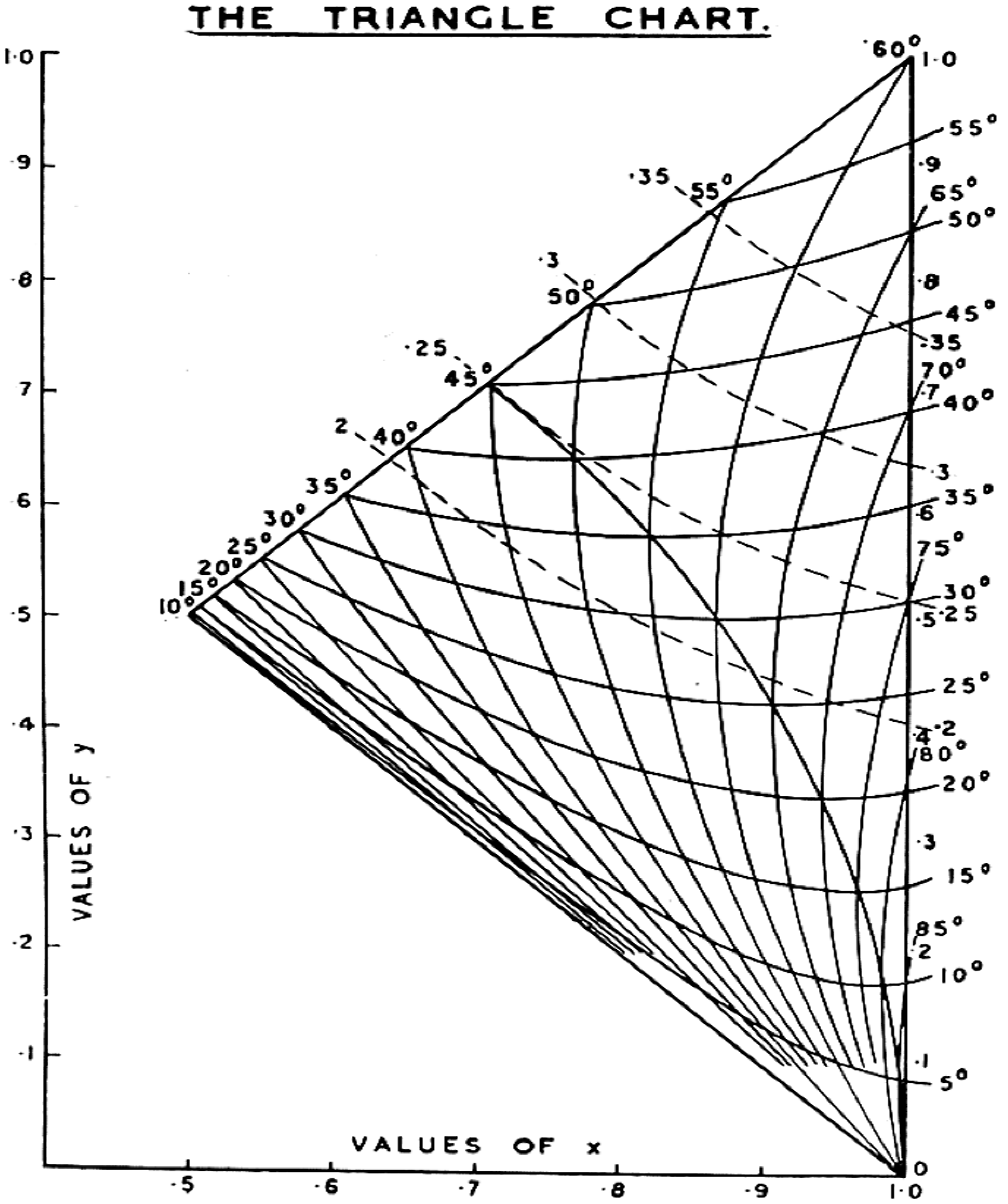

Fig. 215.

FORMULÆ

Definition relations :

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta}.$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Pythagorean relations :

$\cos^2\theta + \sin^2\theta = 1$, $\sec^2\theta = 1 + \tan^2\theta$, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

Special Angles :

	0°	30°	45°	60°	90°
sine . .	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cosine . .	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tangent .	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	Infinite

Negative Angles :

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$.

Complementary ∠s :

$\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$, $\tan(90° - \theta) = \cot\theta$.

Supplementary ∠s :

$\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$.
(For $90° + \theta$, $180° + \theta$, etc., see pp. 118, 119.)

Addition Formulæ :

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}.$$

Double Angles :

$$\sin 2\theta = 2\sin\theta\cos\theta,\ \tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

Triple Angles :

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,\ \cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Half Angles :

$$\sin^2 \tfrac{1}{2}\theta = \tfrac{1}{2}(1 - \cos\theta),\ \cos^2 \tfrac{1}{2}\theta = \tfrac{1}{2}(1 + \cos\theta).$$

t-formulæ ($t = \tan \frac{1}{2}\theta$) :

$$\sin\theta = \frac{2t}{1 + t^2},\ \cos\theta = \frac{1 - t^2}{1 + t^2},\ \tan\theta = \frac{2t}{1 - t^2}.$$

$a \cos\theta + b \sin\theta = r \cos(\theta - \alpha)$; $r = \surd(a^2 + b^2)$, $\tan\alpha = b/a$.

1 radian $= 180^\circ/\pi \overset{\frown}{=} 3438' \overset{\frown}{=} 206265'' \overset{\frown}{=} 57^\circ 17' 45'' \overset{\frown}{=} 57{\cdot}3^\circ$.
$\sin\theta \overset{\frown}{=} \theta$, $\cos\theta \overset{\frown}{=} 1 - \frac{1}{2}\theta^2$, $\tan\theta \overset{\frown}{=} \theta$ if θ radians is a small angle.

Sine formula :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \text{ (radius of circumcircle).}$$

Cosine formula :

$$a^2 = b^2 + c^2 - 2bc \cos A,\ \cos A = (b^2 + c^2 - a^2)/2bc.$$

Double-cosine f. : $a = b \cos C + c \cos B$.

Tangent f. (*Napier's Analogy*) : $\dfrac{\tan\frac{1}{2}(B - C)}{\tan\frac{1}{2}(B + C)} = \dfrac{b - c}{b + c}$.

Cotangent f. : $\cot B + \cot C = \dfrac{a}{b \sin C} = \dfrac{a}{c \sin B}$.

Triangle Area :

$$\triangle = \tfrac{1}{2}bc \sin A = \surd\{s(s - a)(s - b)(s - c).\}$$

$$r = \frac{\triangle}{s} = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}},\ \tan \tfrac{1}{2}A = \frac{r}{s - a}.$$

For other areas and volumes, use the Index.

SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′
0°	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157
1°	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332
2°	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506
3°	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680
4°	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854
5°	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028
6°	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201
7°	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374
8°	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547
9°	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719
10°	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891
11°	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062
12°	2079	2096	2113	2130	2147	2164	2181	2198	2215	2232
13°	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402
14°	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571
15°	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740
16°	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907
17°	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074
18°	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239
19°	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404
20°	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567
21°	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730
22°	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891
23°	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051
24°	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210
25°	4226	4242	4258	4274	4289	4305	4521	4337	4352	4368
26°	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524
27°	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679
28°	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833
29°	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985
30°	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135
31°	5150	5165	5180	6195	5210	5225	5240	5255	5270	5284
32°	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432
33°	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577
34°	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721
35°	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864
36°	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004
37°	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143
38°	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280
39°	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414
40°	6428	6441	6455	6468	6481	6494	6508	6521	6534	6547
41°	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678
42°	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807
43°	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934
44°	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059

SINES

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′
45°	7071	7083	7096	7108	7120	7133	7145	7157	7169	7181
46°	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302
47°	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420
48°	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536
49°	7547	7558	7570	7581	7593	7604	7615	7627	7638	7649
50°	7660	7672	7683	7694	7705	7716	7727	7738	7749	7760
51°	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869
52°	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976
53°	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080
54°	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181
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56°	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377
57°	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471
58°	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563
59°	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652
60°	8660	8669	8678	8686	8695	8704	8712	8721	8729	8738
61°	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821
62°	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902
63°	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980
64°	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056
65°	9063	9070	9078	9085	9092	9100	9107	9114	9121	9128
66°	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198
67°	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265
68°	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330
69°	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391
70°	9397	9403	9409	9415	9421	9426	9432	9438	9444	9449
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72°	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558
73°	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608
74°	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655
75°	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699
76°	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740
77°	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778
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83°	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943
84°	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960
85°	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974
86°	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985
87°	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993
88°	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998
89°	9998	9999	9999	9999	9999	1·0000 nearly	1·0000 nearly	1·0000 nearly	1·0000 nearly	1·0000 nearly

Title [redacted]

Author [redacted]

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